Openly generated Boolean algebras under FRP

Sakaé Fuchino (渕野 昌)

Kobe University

fuchino@diamond.kobe-u.ac.jp

http://kurt.scitec.kobe-u.ac.jp/~fuchino/

(November 28, 2009 (22:46 JST) version)

神戸大学大学院 工学研究科

（旧）新井プロジェクト・プレゼンテーション室

「モデル理論とその周辺」研究集会

November 28, 2009

This presentation is typeset by pLaTeX + beamer class.

Related Papers and Preprints (in chronological order)

[1] S. Fuchino, I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, *Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness*, to appear in Topology and Its Applications.

[2] S. Fuchino, *Left-separated topological spaces under Fodor-type Reflection Principle*, RIMS Kôkyûroku No.1619, (2008), 32–42.

[3] S. Fuchino, *Fodor-type Reflection Principle implies Balogh's theorems under Axiom R*, preprint.

[4] S. Fuchino H. Sakai, L. Soukup and T. Usuba, *More about the Fodor-type Reflection Principle*, in preparation.

[5] S. Fuchino, *Openly generated Boolean algebras and the Fodor-type Reflection Principle*, dedicated to Professor Dr. Sabine Koppelberg on the occasion of her retirement, preprint.

Related Papers and Preprints (in chronological order)

[1] S. Fuchino, I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, *Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness*, to appear in Topology and Its Applications.

[2] S. Fuchino, *Left-separated topological spaces under Fodor-type Reflection Principle*, RIMS Kôkyûroku No.1619, (2008), 32–42.

[3] S. Fuchino, *Fodor-type Reflection Principle implies Balogh's theorems under Axiom R*, preprint.

[4] S. Fuchino H. Sakai, L. Soukup and T. Usuba, *More about the Fodor-type Reflection Principle*, in preparation.

[5] S. Fuchino, *Openly generated Boolean algebras and the Fodor-type Reflection Principle*, dedicated to Professor Dr. Sabine Koppelberg on the occasion of her retirement, preprint.

Related Papers and Preprints (in chronological order)

[1] S. Fuchino, I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, *Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness*, to appear in Topology and Its Applications.

[2] S. Fuchino, *Left-separated topological spaces under Fodor-type Reflection Principle*, RIMS Kôkyûroku No.1619, (2008), 32–42.

[3] S. Fuchino, *Fodor-type Reflection Principle implies Balogh's theorems under Axiom R*, preprint.

[4] S. Fuchino H. Sakai, L. Soukup and T. Usuba, *More about the Fodor-type Reflection Principle*, in preparation.

[5] S. Fuchino, *Openly generated Boolean algebras and the Fodor-type Reflection Principle*, dedicated to Professor Dr. Sabine Koppelberg on the occasion of her retirement, preprint.

Related Papers and Preprints (in chronological order)

[1] S. Fuchino, I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, *Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness*, to appear in Topology and Its Applications.

[2] S. Fuchino, *Left-separated topological spaces under Fodor-type Reflection Principle*, RIMS Kôkyûroku No.1619, (2008), 32–42.

[3] S. Fuchino, *Fodor-type Reflection Principle implies Balogh's theorems under Axiom R*, preprint.

[4] S. Fuchino H. Sakai, L. Soukup and T. Usuba, *More about the Fodor-type Reflection Principle*, in preparation.

[5] S. Fuchino, *Openly generated Boolean algebras and the Fodor-type Reflection Principle*, dedicated to Professor Dr. Sabine Koppelberg on the occasion of her retirement, preprint.

Related Papers and Preprints (in chronological order)

[1] S. Fuchino, I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, *Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness*, to appear in Topology and Its Applications.

[2] S. Fuchino, *Left-separated topological spaces under Fodor-type Reflection Principle*, RIMS Kôkyûroku No.1619, (2008), 32–42.

[3] S. Fuchino, *Fodor-type Reflection Principle implies Balogh's theorems under Axiom R*, preprint.

[4] S. Fuchino H. Sakai, L. Soukup and T. Usuba, *More about the Fodor-type Reflection Principle*, in preparation.

[5] S. Fuchino, *Openly generated Boolean algebras and the Fodor-type Reflection Principle*, dedicated to Professor Dr. Sabine Koppelberg on the occasion of her retirement, preprint.

Related Papers and Preprints (in chronological order)

[1] S. Fuchino, I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, *Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness*, to appear in Topology and Its Applications.

[2] S. Fuchino, *Left-separated topological spaces under Fodor-type Reflection Principle*, RIMS Kôkyûroku No.1619, (2008), 32–42.

[3] S. Fuchino, *Fodor-type Reflection Principle implies Balogh's theorems under Axiom R*, preprint.

[4] S. Fuchino H. Sakai, L. Soukup and T. Usuba, *More about the Fodor-type Reflection Principle*, in preparation.

[5] S. Fuchino, *Openly generated Boolean algebras and the Fodor-type Reflection Principle*, dedicated to Professor Dr. Sabine Koppelberg on the occasion of her retirement, preprint.

Reflection Principles

RP: For cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq [\lambda]^{\aleph_0}$, there is an $I \in [\lambda]^{\aleph_1}$ s.t. $\omega_1 \subseteq I$, $cf(I) = \omega_1$ and $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Axiom R: For cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq [\lambda]^{\aleph_0}$ and ω_1-club $\mathcal{T} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{T}$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Here, $\mathcal{T} \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-club (or "tight and unbounded" in Fleissner's terminology) if

- $\mathcal{T}$ is cofinal in $[X]^{\aleph_1}$ w.r.t. $\subseteq$ and
- for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{T}$ of length ω_1, we have $\bigcup_{\alpha<\omega_1} I_\alpha \in T$.

RP: For cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq [\lambda]^{\aleph_0}$, there is an $I \in [\lambda]^{\aleph_1}$ s.t. $\omega_1 \subseteq I$, $cf(I) = \omega_1$ and $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Axiom R: For cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq [\lambda]^{\aleph_0}$ and ω_1-club $\mathcal{T} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{T}$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Here, $\mathcal{T} \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-club (or "tight and unbounded" in Fleissner's terminology) if

- $\mathcal{T}$ is cofinal in $[X]^{\aleph_1}$ w.r.t. $\subseteq$ and
- for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{T}$ of length ω_1, we have $\bigcup_{\alpha<\omega_1} I_\alpha \in T$.

RP: For cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq [\lambda]^{\aleph_0}$, there is an $I \in [\lambda]^{\aleph_1}$ s.t. $\omega_1 \subseteq I$, $cf(I) = \omega_1$ and $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Axiom R: For cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq [\lambda]^{\aleph_0}$ and ω_1-club $\mathcal{T} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{T}$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Here, $\mathcal{T} \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-club (or "tight and unbounded" in Fleissner's terminology) if

- $\mathcal{T}$ is cofinal in $[X]^{\aleph_1}$ w.r.t. $\subseteq$ and
- for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{T}$ of length ω_1, we have $\bigcup_{\alpha<\omega_1} I_\alpha \in T$.

RP: For cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq [\lambda]^{\aleph_0}$, there is an $I \in [\lambda]^{\aleph_1}$ s.t. $\omega_1 \subseteq I$, $cf(I) = \omega_1$ and $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Axiom R: For cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq [\lambda]^{\aleph_0}$ and ω_1-club $\mathcal{T} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{T}$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Here, $\mathcal{T} \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-club (or "tight and unbounded" in Fleissner's terminology) if

- $\mathcal{T}$ is cofinal in $[X]^{\aleph_1}$ w.r.t. $\subseteq$ and
- for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{T}$ of length ω_1, we have $\bigcup_{\alpha<\omega_1} I_\alpha \in T$.

MM $\Rightarrow$ $\mathrm{MA}^+(\sigma\text{-closed})$ $\Rightarrow$ Axiom R $\Rightarrow$ RP

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

$$\mathrm{MM} \Rightarrow \mathrm{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP}$$

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

$\text{MM} \Rightarrow \text{MA}^{+}(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \text{RP}$

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

MM $\Rightarrow$ $\mathrm{MA}^+(\sigma$-closed$)$ $\Rightarrow$ Axiom R $\Rightarrow$ RP

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

$\mathrm{MM} \Rightarrow \mathrm{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP}$

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

MM $\Rightarrow$ $\mathrm{MA}^+(\sigma\text{-closed})$ $\Rightarrow$ Axiom R $\Rightarrow$ RP

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

MM $\Rightarrow$ $\mathrm{MA}^{+}(\sigma\text{-closed})$ $\Rightarrow$ Axiom R $\Rightarrow$ RP

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

$\text{MM} \Rightarrow \mathrm{MA}^{+}(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP}$

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

Reflection Principles

$\text{MM} \Rightarrow \text{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \text{RP}$

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

MM $\Rightarrow$ $\mathrm{MA}^+(\sigma\text{-closed})$ $\Rightarrow$ Axiom R $\Rightarrow$ RP

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

MM $\Rightarrow$ MA^+(σ-closed) $\Rightarrow$ Axiom R $\Rightarrow$ RP

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

Reflection Principles

MM $\Rightarrow$ MA^+(σ-closed) $\Rightarrow$ Axiom R $\Rightarrow$ RP

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

MM $\Rightarrow$ $\mathrm{MA}^+(\sigma\text{-closed})$ $\Rightarrow$ Axiom R $\Rightarrow$ RP

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

Fleissner's Theorem on left-separated spaces

Theorem 1 (W. Fleissner 1986)

Assume Axiom R. Suppose that X is a T_1-space with a point countable base. If X is not left-separated then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not left-separated.

- A topological space X is left-separated if there is a well-ordering $<$ of X s.t. every initial segment w.r.t. $<$ is a closed subset of X.

Fleissner's Theorem on left-separated spaces

Theorem 1 (W. Fleissner 1986)

Assume Axiom R. Suppose that X is a T_1-space with a point countable base. If X is not left-separated then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not left-separated.

- A topological space X is left-separated if there is a well-ordering $<$ of X s.t. every initial segment w.r.t. $<$ is a closed subset of X.

Fleissner's Theorem on coloring number of graphs

Theorem 2 (W. Fleissner 1986)

Assume Axiom R. If a graph (V, E) *has coloring number* $\geq \aleph_1$ *then there is an infinite subgraph of* (V, E) *of cardinality* $\aleph_1$ *with coloring number* $\aleph_1$.

- For a graph (V, E) the coloring number of (V, E) is the minimal cardinal μ s.t.

  there is a well-ordering $\prec$ of V s.t., for every $v \in V$, the set $\{u \in V \,:\, u \prec v,\ \{u, v\} \in E\}$ has cardinality $< \mu$.

Fleissner's Theorem on coloring number of graphs

Theorem 2 (W. Fleissner 1986)

Assume Axiom R. If a graph (V, E) *has coloring number* $\geq \aleph_1$ *then there is an infinite subgraph of* (V, E) *of cardinality* $\aleph_1$ *with coloring number* $\aleph_1$.

- For a graph (V, E) the coloring number of (V, E) is the minimal cardinal μ s.t.

  there is a well-ordering $\prec$ of V s.t., for every $v \in V$, the set $\{u \in V \,:\, u \prec v,\ \{u, v\} \in E\}$ has cardinality $< \mu$.

Balogh's reflection theorem on metrizability

Theorem 3 (Z. Balogh 2002)

Assume Axiom R. Suppose that X is locally countably compact. If X is not metrizable then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not metrizable.

- A topological space X is countably compact if any countable open cover of X has a finite subcover.
- A topological space X is locally countably compact if any point of X has a neighborhood which is countably compact.

Balogh's reflection theorem on metrizability

Theorem 3 (Z. Balogh 2002)

Assume Axiom R. Suppose that X is locally countably compact. If X is not metrizable then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not metrizable.

- A topological space X is countably compact if any countable open cover of X has a finite subcover.

- A topological space X is locally countably compact if any point of X has a neighborhood which is countably compact.

Balogh's reflection theorem on metrizability

Theorem 3 (Z. Balogh 2002)

Assume Axiom R. Suppose that X is locally countably compact. If X is not metrizable then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not metrizable.

- A topological space X is countably compact if any countable open cover of X has a finite subcover.

- A topological space X is locally countably compact if any point of X has a neighborhood which is countably compact.

Balogh's reflection theorem on metrizability

Theorem 3 (Z. Balogh 2002)

Assume Axiom R. Suppose that X is locally countably compact. If X is not metrizable then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not metrizable.

- A topological space X is countably compact if any countable open cover of X has a finite subcover.

- A topological space X is locally countably compact if any point of X has a neighborhood which is countably compact.

Fodor-type reflection principle (FRP)

The 'mathematical' reflection theorems mentioned above are actually consequences of the following combinatorial principle which is much weaker than Axiom R (even weaker than RP):

FRP: For any regular cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq E^{\lambda}_{\omega}$ and mapping $g : S \to [\lambda]^{\leq\aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ s.t.

- $\mathrm{cf}(I) = \omega_1$;
- $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- for any regressive $f : S \cap I \to \lambda$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \lambda$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

The 'mathematical' reflection theorems mentioned above are actually consequences of the following combinatorial principle which is much weaker than Axiom R (even weaker than RP):

FRP: For any regular cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq E^\lambda_\omega$ and mapping $g : S \to [\lambda]^{\leq\aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ s.t.

- $\mathrm{cf}(I) = \omega_1$;
- $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- for any regressive $f : S \cap I \to \lambda$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \lambda$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

The ‘mathematical’ reflection theorems mentioned above are actually consequences of the following combinatorial principle which is much weaker than Axiom R (even weaker than RP):

FRP: For any regular cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq E^\lambda_\omega$ and mapping $g : S \to [\lambda]^{\leq\aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ s.t.

- $\mathrm{cf}(I) = \omega_1$;
- $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- for any regressive $f : S \cap I \to \lambda$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \lambda$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension (this is not the case for RP).

$$\mathrm{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{FRP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP: For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

Fodor-type reflection principle (FRP)

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension (this is not the case for RP).

$$\mathrm{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP} \underset{\Rightarrow}{\not\Leftarrow} \mathrm{FRP} \underset{\Rightarrow}{\not\Leftarrow} \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP: For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension (this is not the case for RP).

$$\mathrm{MA}^{+}(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP} \underset{\Rightarrow}{\not\Leftarrow} \mathrm{FRP} \underset{\Rightarrow}{\not\Leftarrow} \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP: For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension (this is not the case for RP).

$$\mathrm{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{FRP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP: For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension (this is not the case for RP).

$$\mathrm{MA}^{+}(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP} \underset{\Rightarrow}{\not\Leftarrow} \mathrm{FRP} \underset{\Rightarrow}{\not\Leftarrow} \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP: For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension (this is not the case for RP).

$$\mathrm{MA}^{+}(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP} \underset{\Rightarrow}{\not\Leftarrow} \mathrm{FRP} \underset{\Rightarrow}{\not\Leftarrow} \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP: For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension
  (this is not the case for RP).

$$\mathrm{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{FRP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP:
  For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension (this is not the case for RP).

$$\mathrm{MA}^{+}(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{FRP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP: For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension (this is not the case for RP).

$$\mathrm{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{FRP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP: For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension
  (this is not the case for RP).

$$\mathrm{MA}^+(\sigma\text{-closed}) \;\Rightarrow\; \text{Axiom R} \;\Rightarrow\; \mathrm{RP} \;\overset{\not\Leftarrow}{\Rightarrow}\; \mathrm{FRP} \;\overset{\not\Leftarrow}{\Rightarrow}\; \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP:
  For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension (this is not the case for RP).

$$\mathrm{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP} \underset{\Rightarrow}{\not\Leftarrow} \mathrm{FRP} \underset{\Rightarrow}{\not\Leftarrow} \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP: For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

"Mathematical" characterizations of FRP

In F., Sakai, Soukup and Usuba [4] it is proved that theorems mentioned on the last slide are all equivalent to FRP over ZFC.

Corollary 4

FRP is equivalent to each of the following assertions over ZFC:

(A) *For every locally separable countably tight topological space X, if X is not meta-Lindelöf, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf.*

(B) *For every locally countably compact topological space X, if X is not metrizable, then there is a subspaces of X of cardinality $\leq \aleph_1$ which is not metrizable.*

(C) *For every T_1-space X with a point countable base, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(C') *For every metrizable space X, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(D) *For any graph $G = \langle G, \mathcal{E} \rangle$ if the coloring number of G is uncountable, then there is $I \in [G]^{\aleph_1}$ s.t. the coloring number of $G \restriction I$ is uncountable.*

"Mathematical" characterizations of FRP

In F., Sakai, Soukup and Usuba [4] it is proved that theorems mentioned on the last slide are all equivalent to FRP over ZFC.

Corollary 4

FRP *is equivalent to each of the following assertions over* ZFC*:*

(A) *For every locally separable countably tight topological space X, if X is not meta-Lindelöf, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf.*

(B) *For every locally countably compact topological space X, if X is not metrizable, then there is a subspaces of X of cardinality $\leq \aleph_1$ which is not metrizable.*

(C) *For every T_1-space X with a point countable base, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(C') *For every metrizable space X, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(D) *For any graph $G = \langle G, \mathcal{E} \rangle$ if the coloring number of G is uncountable, then there is $I \in [G]^{\aleph_1}$ s.t. the coloring number of $G \restriction I$ is uncountable.*

In F., Sakai, Soukup and Usuba [4] it is proved that theorems mentioned on the last slide are all equivalent to FRP over ZFC.

Corollary 4

FRP *is equivalent to each of the following assertions over* ZFC*:*

(A) *For every locally separable countably tight topological space X, if X is not meta-Lindelöf, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf.*

(B) *For every locally countably compact topological space X, if X is not metrizable, then there is a subspaces of X of cardinality $\leq \aleph_1$ which is not metrizable.*

(C) *For every T_1-space X with a point countable base, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(C') *For every metrizable space X, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(D) *For any graph $G = \langle G, \mathcal{E} \rangle$ if the coloring number of G is uncountable, then there is $I \in [G]^{\aleph_1}$ s.t. the coloring number of $G \restriction I$ is uncountable.*

"Mathematical" characterizations of FRP

In F., Sakai, Soukup and Usuba [4] it is proved that theorems mentioned on the last slide are all equivalent to FRP over ZFC.

Corollary 4

FRP *is equivalent to each of the following assertions over* ZFC*:*

(A) *For every locally separable countably tight topological space X, if X is not meta-Lindelöf, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf.*

(B) *For every locally countably compact topological space X, if X is not metrizable, then there is a subspaces of X of cardinality $\leq \aleph_1$ which is not metrizable.*

(C) *For every T_1-space X with a point countable base, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(C') *For every metrizable space X, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(D) *For any graph $G = \langle G, \mathcal{E} \rangle$ if the coloring number of G is uncountable, then there is $I \in [G]^{\aleph_1}$ s.t. the coloring number of $G \restriction I$ is uncountable.*

In F., Sakai, Soukup and Usuba [4] it is proved that theorems mentioned on the last slide are all equivalent to FRP over ZFC.

Corollary 4

FRP is equivalent to each of the following assertions over ZFC:

(A) *For every locally separable countably tight topological space X, if X is not meta-Lindelöf, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf.*

(B) *For every locally countably compact topological space X, if X is not metrizable, then there is a subspaces of X of cardinality $\leq \aleph_1$ which is not metrizable.*

(C) *For every T_1-space X with a point countable base, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(C') *For every metrizable space X, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(D) *For any graph $G = \langle G, \mathcal{E} \rangle$ if the coloring number of G is uncountable, then there is $I \in [G]^{\aleph_1}$ s.t. the coloring number of $G \restriction I$ is uncountable.*

In F., Sakai, Soukup and Usuba [4] it is proved that theorems mentioned on the last slide are all equivalent to FRP over ZFC.

Corollary 4

FRP is equivalent to each of the following assertions over ZFC:

(A) *For every locally separable countably tight topological space X, if X is not meta-Lindelöf, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf.*

(B) *For every locally countably compact topological space X, if X is not metrizable, then there is a subspaces of X of cardinality $\leq \aleph_1$ which is not metrizable.*

(C) *For every T_1-space X with a point countable base, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(C') *For every metrizable space X, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(D) *For any graph $G = \langle G, \mathcal{E} \rangle$ if the coloring number of G is uncountable, then there is $I \in [G]^{\aleph_1}$ s.t. the coloring number of $G \upharpoonright I$ is uncountable.*

“Mathematical” characterizations of FRP

In F., Sakai, Soukup and Usuba [4] it is proved that theorems mentioned on the last slide are all equivalent to FRP over ZFC.

Corollary 4

FRP *is equivalent to each of the following assertions over* ZFC*:*

(A) *For every locally separable countably tight topological space X, if X is not meta-Lindelöf, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf.*

(B) *For every locally countably compact topological space X, if X is not metrizable, then there is a subspaces of X of cardinality $\leq \aleph_1$ which is not metrizable.*

(C) *For every T_1-space X with a point countable base, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(C') *For every metrizable space X, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(D) *For any graph $G = \langle G, \mathcal{E} \rangle$ if the coloring number of G is uncountable, then there is $I \in [G]^{\aleph_1}$ s.t. the coloring number of $G \restriction I$ is uncountable.*

Seeming extension of FRP

Lemma 5 (FRP)

Suppose that $\lambda > \aleph_1$ is a regular cardinal. Then, for any mapping $g : S \to [\lambda]^{\leq\aleph_0}$ on a stationary $S \subseteq E^\lambda_\omega$ and closed unbounded $\mathcal{C} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{C}$ as in the definition of FRP.

Proof. Suppose that λ, g, S, $\mathcal{C}$ are as above. Let sk be the canonical Skolem-hull operator on $\langle \mathcal{H}(\theta), \in, g, \mathcal{C}, \ldots, \sqsubseteq \rangle$ for a sufficiently large regular θ, where $\sqsubseteq$ is a well-ordering on $\mathcal{H}(\theta)$.

Let $C^* = \{\alpha < \kappa : sk(\alpha) \cap \kappa = \alpha\}$ and let $h : \kappa \to \kappa$ be defined by $h(\alpha) = \min C^* \setminus \alpha$ for $\alpha \in \kappa$.

Let $g' : S \to [\kappa]^{\leq\aleph_0}$ be defined by $g'(\alpha) = g(\alpha) \cup \{h(\sup g(\alpha))\}$ for $\alpha \in S$.

By FRP there is $I_0 \in [\kappa]^{\aleph_1}$ as in the definition of FRP for S and g'. Then, since C^* is closed, we have $\sup(I_0) \in C^*$. Let $I \in [\kappa]^{\aleph_1}$ be s.t. $I_0 \subseteq I \subseteq \sup(I_0)$ and $sk(I) \cap \kappa = I$. Then $I \in \mathcal{C}$ since $I = \bigcup(\mathcal{C} \cap sk(I))$ and I is as desired. □

Seeming extension of FRP

Lemma 5 (FRP)

Suppose that $\lambda > \aleph_1$ is a regular cardinal. Then, for any mapping $g : S \to [\lambda]^{\leq\aleph_0}$ on a stationary $S \subseteq E^\lambda_\omega$ and closed unbounded $\mathcal{C} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{C}$ as in the definition of FRP.

Proof. Suppose that λ, g, S, $\mathcal{C}$ are as above. Let sk be the canonical Skolem-hull operator on $\langle \mathcal{H}(\theta), \in, g, \mathcal{C}, \ldots, \sqsubseteq \rangle$ for a sufficiently large regular θ, where $\sqsubseteq$ is a well-ordering on $\mathcal{H}(\theta)$.
Let $C^* = \{\alpha < \kappa : sk(\alpha) \cap \kappa = \alpha\}$ and let $h : \kappa \to \kappa$ be defined by $h(\alpha) = \min C^* \setminus \alpha$ for $\alpha \in \kappa$.
Let $g' : S \to [\kappa]^{\leq\aleph_0}$ be defined by $g'(\alpha) = g(\alpha) \cup \{h(\sup g(\alpha))\}$ for $\alpha \in S$.
By FRP there is $I_0 \in [\kappa]^{\aleph_1}$ as in the definition of FRP for S and g'. Then, since C^* is closed, we have $\sup(I_0) \in C^*$. Let $I \in [\kappa]^{\aleph_1}$ be s.t. $I_0 \subseteq I \subseteq \sup(I_0)$ and $sk(I) \cap \kappa = I$. Then $I \in \mathcal{C}$ since $I = \bigcup(\mathcal{C} \cap sk(I))$ and I is as desired. □

Seeming extension of FRP

Lemma 5 (FRP)

Suppose that $\lambda > \aleph_1$ is a regular cardinal. Then, for any mapping $g : S \to [\lambda]^{\leq\aleph_0}$ on a stationary $S \subseteq E^\lambda_\omega$ and closed unbounded $\mathcal{C} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{C}$ as in the definition of FRP.

Proof. Suppose that λ, g, S, $\mathcal{C}$ are as above. Let sk be the canonical Skolem-hull operator on $\langle \mathcal{H}(\theta), \in, g, \mathcal{C}, \ldots, \sqsubseteq \rangle$ for a sufficiently large regular θ, where $\sqsubseteq$ is a well-ordering on $\mathcal{H}(\theta)$. Let $C^* = \{\alpha < \kappa : sk(\alpha) \cap \kappa = \alpha\}$ and let $h : \kappa \to \kappa$ be defined by $h(\alpha) = \min C^* \setminus \alpha$ for $\alpha \in \kappa$.

Let $g' : S \to [\kappa]^{\leq\aleph_0}$ be defined by $g'(\alpha) = g(\alpha) \cup \{h(\sup g(\alpha))\}$ for $\alpha \in S$.

By FRP there is $I_0 \in [\kappa]^{\aleph_1}$ as in the definition of FRP for S and g'. Then, since C^* is closed, we have $\sup(I_0) \in C^*$. Let $I \in [\kappa]^{\aleph_1}$ be s.t. $I_0 \subseteq I \subseteq \sup(I_0)$ and $sk(I) \cap \kappa = I$. Then $I \in \mathcal{C}$ since $I = \bigcup(\mathcal{C} \cap sk(I))$ and I is as desired. □

Lemma 5 (FRP)

Suppose that $\lambda > \aleph_1$ is a regular cardinal. Then, for any mapping $g : S \to [\lambda]^{\leq\aleph_0}$ on a stationary $S \subseteq E^\lambda_\omega$ and closed unbounded $\mathcal{C} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{C}$ as in the definition of FRP.

Proof. Suppose that λ, g, S, $\mathcal{C}$ are as above. Let sk be the canonical Skolem-hull operator on $\langle \mathcal{H}(\theta), \in, g, \mathcal{C}, \ldots, \sqsubseteq \rangle$ for a sufficiently large regular θ, where $\sqsubseteq$ is a well-ordering on $\mathcal{H}(\theta)$.
Let $C^* = \{\alpha < \kappa : sk(\alpha) \cap \kappa = \alpha\}$ and let $h : \kappa \to \kappa$ be defined by $h(\alpha) = \min C^* \setminus \alpha$ for $\alpha \in \kappa$.
Let $g' : S \to [\kappa]^{\leq\aleph_0}$ be defined by $g'(\alpha) = g(\alpha) \cup \{h(\sup g(\alpha))\}$ for $\alpha \in S$.
By FRP there is $I_0 \in [\kappa]^{\aleph_1}$ as in the definition of FRP for S and g'. Then, since C^* is closed, we have $\sup(I_0) \in C^*$. Let $I \in [\kappa]^{\aleph_1}$ be s.t. $I_0 \subseteq I \subseteq \sup(I_0)$ and $sk(I) \cap \kappa = I$. Then $I \in \mathcal{C}$ since $I = \bigcup(\mathcal{C} \cap sk(I))$ and I is as desired. □

Seeming extension of FRP

Lemma 5 (FRP)

Suppose that $\lambda > \aleph_1$ is a regular cardinal. Then, for any mapping $g : S \to [\lambda]^{\leq\aleph_0}$ on a stationary $S \subseteq E^\lambda_\omega$ and closed unbounded $\mathcal{C} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{C}$ as in the definition of FRP.

Proof. Suppose that λ, g, S, $\mathcal{C}$ are as above. Let sk be the canonical Skolem-hull operator on $\langle \mathcal{H}(\theta), \in, g, \mathcal{C}, \ldots, \sqsubseteq\rangle$ for a sufficiently large regular θ, where $\sqsubseteq$ is a well-ordering on $\mathcal{H}(\theta)$.
Let $C^* = \{\alpha < \kappa \,:\, sk(\alpha) \cap \kappa = \alpha\}$ and let $h : \kappa \to \kappa$ be defined by $h(\alpha) = \min C^* \setminus \alpha$ for $\alpha \in \kappa$.
Let $g' : S \to [\kappa]^{\leq\aleph_0}$ be defined by $g'(\alpha) = g(\alpha) \cup \{h(\sup g(\alpha))\}$ for $\alpha \in S$.
By FRP there is $I_0 \in [\kappa]^{\aleph_1}$ as in the definition of FRP for S and g'. Then, since C^* is closed, we have $\sup(I_0) \in C^*$. Let $I \in [\kappa]^{\aleph_1}$ be s.t. $I_0 \subseteq I \subseteq \sup(I_0)$ and $sk(I) \cap \kappa = I$. Then $I \in \mathcal{C}$ since $I = \bigcup(\mathcal{C} \cap sk(I))$ and I is as desired. □

Seeming extension of FRP

Lemma 5 (FRP)

Suppose that $\lambda > \aleph_1$ is a regular cardinal. Then, for any mapping $g : S \to [\lambda]^{\leq\aleph_0}$ on a stationary $S \subseteq E^\lambda_\omega$ and closed unbounded $\mathcal{C} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{C}$ as in the definition of FRP.

Proof. Suppose that λ, g, S, $\mathcal{C}$ are as above. Let sk be the canonical Skolem-hull operator on $\langle\mathcal{H}(\theta), \in, g, \mathcal{C}, \ldots, \sqsubseteq\rangle$ for a sufficiently large regular θ, where $\sqsubseteq$ is a well-ordering on $\mathcal{H}(\theta)$.
Let $C^* = \{\alpha < \kappa : sk(\alpha) \cap \kappa = \alpha\}$ and let $h : \kappa \to \kappa$ be defined by $h(\alpha) = \min C^* \setminus \alpha$ for $\alpha \in \kappa$.
Let $g' : S \to [\kappa]^{\leq\aleph_0}$ be defined by $g'(\alpha) = g(\alpha) \cup \{h(\sup g(\alpha))\}$ for $\alpha \in S$.
By FRP there is $I_0 \in [\kappa]^{\aleph_1}$ as in the definition of FRP for S and g'. Then, since C^* is closed, we have $\sup(I_0) \in C^*$. Let $I \in [\kappa]^{\aleph_1}$ be s.t. $I_0 \subseteq I \subseteq \sup(I_0)$ and $sk(I) \cap \kappa = I$. Then $I \in \mathcal{C}$ since $I = \bigcup(\mathcal{C} \cap sk(I))$ and I is as desired. □

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B \text{ and } A \text{ is openly generated}\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B \text{ and } A \text{ is openly generated}\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B \text{ and } A \text{ is openly generated}\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B \text{ and } A \text{ is openly generated}\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B \text{ and } A \text{ is openly generated}\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B \text{ and } A \text{ is openly generated}\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B \text{ and } A \text{ is openly generated}\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_\sigma B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B$ and A is openly generated$\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_\sigma B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B$ and A is openly generated$\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B \text{ and } A \text{ is openly generated}\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B$ and A is openly generated$\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_\sigma B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B$ and A is openly generated$\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B$ and A is openly generated$\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\rm rc} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_\sigma B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\rm rc} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B \text{ and } A \text{ is openly generated}\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\mathrm{rc}} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B \text{ and } A \text{ is openly generated}\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

Independence of Theorem 6

By the following Theorem, the assertion of Theorem 6 on the last slide is independent:

Theorem 7 (F. 1993)

For a regular cardinal κ, if there is a non-reflecting stationary set $E \subseteq E^{\kappa}_{\omega}$, there is a Boolean algebra B with a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ s.t. B_α is a free Boolean algebra for all $\alpha < \kappa$ and $\{\alpha < \kappa : B_\alpha \leq_{\neg\mathrm{rc}} B\} = E$.

B in the theorem above is κ-openly generated (it is even κ-free and $\mathcal{L}_{\infty,\kappa}$-free).

But B is not openly generated (by the theorem on the next slide).

Independence of Theorem 6

By the following Theorem, the assertion of Theorem 6 on the last slide is independent:

Theorem 7 (F. 1993)

For a regular cardinal κ, if there is a non-reflecting stationary set $E \subseteq E^{\kappa}_{\omega}$, there is a Boolean algebra B with a filtration $\langle B_\alpha : \alpha < \kappa\rangle$ s.t. B_α is a free Boolean algebra for all $\alpha < \kappa$ and $\{\alpha < \kappa : B_\alpha \leq_{\neg\mathrm{rc}} B\} = E$.

B in the theorem above is κ-openly generated (it is even κ-free and $\mathcal{L}_{\infty,\kappa}$-free).

But B is not openly generated (by the theorem on the next slide).

By the following Theorem, the assertion of Theorem 6 on the last slide is independent:

Theorem 7 (F. 1993)

For a regular cardinal κ, if there is a non-reflecting stationary set $E \subseteq E^\kappa_\omega$, there is a Boolean algebra B with a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ s.t. B_α is a free Boolean algebra for all $\alpha < \kappa$ and $\{\alpha < \kappa : B_\alpha \leq_{\neg \mathrm{rc}} B\} = E$.

B in the theorem above is κ-openly generated (it is even κ-free and $\mathcal{L}_{\infty,\kappa}$-free).

But B is not openly generated (by the theorem on the next slide).

Independence of Theorem 6

By the following Theorem, the assertion of Theorem 6 on the last slide is independent:

Theorem 7 (F. 1993)

For a regular cardinal κ, if there is a non-reflecting stationary set $E \subseteq E^\kappa_\omega$, there is a Boolean algebra B with a filtration $\langle B_\alpha : \alpha < \kappa\rangle$ s.t. B_α is a free Boolean algebra for all $\alpha < \kappa$ and $\{\alpha < \kappa : B_\alpha \leq_{\neg\mathrm{rc}} B\} = E$.

B in the theorem above is κ-openly generated (it is even κ-free and $\mathcal{L}_{\infty,\kappa}$-free).

But B is not openly generated (by the theorem on the next slide).

Independence of Theorem 6

By the following Theorem, the assertion of Theorem 6 on the last slide is independent:

Theorem 7 (F. 1993)

For a regular cardinal κ, if there is a non-reflecting stationary set $E \subseteq E^{\kappa}_{\omega}$, there is a Boolean algebra B with a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ s.t. B_α is a free Boolean algebra for all $\alpha < \kappa$ and $\{\alpha < \kappa : B_\alpha \leq_{\neg\mathrm{rc}} B\} = E$.

B in the theorem above is κ-openly generated (it is even κ-free and $\mathcal{L}_{\infty,\kappa}$-free).

But B is not openly generated (by the theorem on the next slide).

Characterizations of openly generated Boolean algebras

Theorem 8

For a Boolean algebra B, the following are equivalent:

- *B is openly generated.*
- (L. Heindorf and L.B. Shapiro 1994) *There is a FN-mapping $f : B \to [B]^{<\aleph_0}$ (i.e. such f that, for any $a, b \in B$ with $a \leq b$ there is $c \in f(a) \cap f(b)$ s.t. $a \leq c \leq b$).*
- (F. 1994) *For any sufficiently large regular cardinal θ, and any (countable) $M \prec \mathcal{H}(\theta)$ with $B \in M$, we have $B \cap M \leq_{\mathrm{rc}} B$.*
- (F. 1994) *For any σ-closed poset $\mathbb{P}$ forcing $|B| \leq \aleph_1$, we have $\Vdash_{\mathbb{P}}$ "B is projective Boolean algebra".*

Characterizations of openly generated Boolean algebras

Theorem 8

For a Boolean algebra B, the following are equivalent:

- *B is openly generated.*
- (L. Heindorf and L.B. Shapiro 1994) *There is a FN-mapping $f : B \to [B]^{<\aleph_0}$ (i.e. such f that, for any $a, b \in B$ with $a \leq b$ there is $c \in f(a) \cap f(b)$ s.t. $a \leq c \leq b$).*
- (F. 1994) *For any sufficiently large regular cardinal θ, and any (countable) $M \prec \mathcal{H}(\theta)$ with $B \in M$, we have $B \cap M \leq_{\mathrm{rc}} B$.*
- (F. 1994) *For any σ-closed poset $\mathbb{P}$ forcing $|B| \leq \aleph_1$, we have $\Vdash_{\mathbb{P}}$ "B is projective Boolean algebra".*

Characterizations of openly generated Boolean algebras

Theorem 8

For a Boolean algebra B, the following are equivalent:

- B *is openly generated.*
- (L. Heindorf and L.B. Shapiro 1994) *There is a FN-mapping* $f : B \to [B]^{<\aleph_0}$ *(i.e. such* f *that, for any* $a, b \in B$ *with* $a \le b$ *there is* $c \in f(a) \cap f(b)$ *s.t.* $a \le c \le b$*).*
- (F. 1994) *For any sufficiently large regular cardinal* θ*, and any (countable)* $M \prec \mathcal{H}(\theta)$ *with* $B \in M$*, we have* $B \cap M \le_{\rm rc} B$*.*
- (F. 1994) *For any* σ*-closed poset* $\mathbb{P}$ *forcing* $|B| \le \aleph_1$*, we have* $\Vdash_{\mathbb{P}}$ "B *is projective Boolean algebra*"*.*

Characterizations of openly generated Boolean algebras

Theorem 8

For a Boolean algebra B, the following are equivalent:

- B *is openly generated.*
- (L. Heindorf and L.B. Shapiro 1994) *There is a FN-mapping* $f : B \to [B]^{<\aleph_0}$ *(i.e. such* f *that, for any* $a, b \in B$ *with* $a \leq b$ *there is* $c \in f(a) \cap f(b)$ *s.t.* $a \leq c \leq b$*).*
- (F. 1994) *For any sufficiently large regular cardinal* θ*, and any (countable)* $M \prec \mathcal{H}(\theta)$ *with* $B \in M$*, we have* $B \cap M \leq_{\mathrm{rc}} B$*.*
- (F. 1994) *For any* σ*-closed poset* $\mathbb{P}$ *forcing* $|B| \leq \aleph_1$*, we have* $\Vdash_{\mathbb{P}}$ "B *is projective Boolean algebra*"*.*

Characterizations of openly generated Boolean algebras

Theorem 8

For a Boolean algebra B, the following are equivalent:

- *B is openly generated.*
- (L. Heindorf and L.B. Shapiro 1994) *There is a FN-mapping $f : B \to [B]^{<\aleph_0}$ (i.e. such f that, for any $a, b \in B$ with $a \leq b$ there is $c \in f(a) \cap f(b)$ s.t. $a \leq c \leq b$).*
- (F. 1994) *For any sufficiently large regular cardinal θ, and any (countable) $M \prec \mathcal{H}(\theta)$ with $B \in M$, we have $B \cap M \leq_{\mathrm{rc}} B$.*
- (F. 1994) *For any σ-closed poset $\mathbb{P}$ forcing $|B| \leq \aleph_1$, we have $\Vdash_{\mathbb{P}}$ " B is projective Boolean algebra".*

Characterizations of κ-openly generatedness

Theorem 9 (F. 1995)

For a Boolean algebra B and regular $\kappa > \aleph_1$, the following are equivalent:

- *B is κ-openly generated*
- *B has the c.c.c., satisfies the Bockstein Separation Property, "μ-stable" for all $\aleph_0 < \mu < \kappa$ and $\{C \in [B]^{<\kappa} : C \leq B \text{ and } C \text{ is openly generated}\}$ is cofinal in $[B]^{<\kappa}$*
- *for any sufficiently large regular θ and for any $M \prec \mathcal{H}(\theta)$ s.t. $B \in M$, $\omega_1 \subseteq M$ and $|M| < \kappa$, we have that $B \cap M \leq_\sigma B$ and $B \cap M$ is openly generated.*

Corollary 10

For a regular cardinal κ, a κ-openly generated Boolean algebra B of cardinality κ has a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ consisting of openly generated Boolean algebras s.t. $B_\alpha \leq_\sigma B$ for all $\alpha < \kappa$.

Characterizations of κ-openly generatedness

Theorem 9 (F. 1995)

For a Boolean algebra B and regular $\kappa > \aleph_1$, the following are equivalent:

- *B is κ-openly generated*
- *B has the c.c.c., satisfies the Bockstein Separation Property, "μ-stable" for all $\aleph_0 < \mu < \kappa$ and $\{C \in [B]^{<\kappa} : C \leq B \text{ and } C \text{ is openly generated}\}$ is cofinal in $[B]^{<\kappa}$*
- *for any sufficiently large regular θ and for any $M \prec \mathcal{H}(\theta)$ s.t. $B \in M$, $\omega_1 \subseteq M$ and $|M| < \kappa$, we have that $B \cap M \leq_\sigma B$ and $B \cap M$ is openly generated.*

Corollary 10

For a regular cardinal κ, a κ-openly generated Boolean algebra B of cardinality κ has a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ consisting of openly generated Boolean algebras s.t. $B_\alpha \leq_\sigma B$ for all $\alpha < \kappa$.

Characterizations of κ-openly generatedness

Theorem 9 (F. 1995)

For a Boolean algebra B and regular $\kappa > \aleph_1$, the following are equivalent:

- *B is κ-openly generated*
- *B has the c.c.c., satisfies the Bockstein Separation Property, "μ-stable" for all $\aleph_0 < \mu < \kappa$ and $\{C \in [B]^{<\kappa} : C \leq B \text{ and } C \text{ is openly generated}\}$ is cofinal in $[B]^{<\kappa}$*
- *for any sufficiently large regular θ and for any $M \prec \mathcal{H}(\theta)$ s.t. $B \in M$, $\omega_1 \subseteq M$ and $|M| < \kappa$, we have that $B \cap M \leq_\sigma B$ and $B \cap M$ is openly generated.*

Corollary 10

For a regular cardinal κ, a κ-openly generated Boolean algebra B of cardinality κ has a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ consisting of openly generated Boolean algebras s.t. $B_\alpha \leq_\sigma B$ for all $\alpha < \kappa$.

Characterizations of κ-openly generatedness

Theorem 9 (F. 1995)

For a Boolean algebra B and regular $\kappa > \aleph_1$, the following are equivalent:

- *B is κ-openly generated*
- *B has the c.c.c., satisfies the Bockstein Separation Property, "μ-stable" for all $\aleph_0 < \mu < \kappa$ and $\{C \in [B]^{<\kappa} : C \leq B \text{ and } C \text{ is openly generated}\}$ is cofinal in $[B]^{<\kappa}$*
- *for any sufficiently large regular θ and for any $M \prec \mathcal{H}(\theta)$ s.t. $B \in M$, $\omega_1 \subseteq M$ and $|M| < \kappa$, we have that $B \cap M \leq_\sigma B$ and $B \cap M$ is openly generated.*

Corollary 10

For a regular cardinal κ, a κ-openly generated Boolean algebra B of cardinality κ has a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ consisting of openly generated Boolean algebras s.t. $B_\alpha \leq_\sigma B$ for all $\alpha < \kappa$.

Theorem 9 (F. 1995)

For a Boolean algebra B and regular $\kappa > \aleph_1$, the following are equivalent:

- *B is κ-openly generated*
- *B has the c.c.c., satisfies the Bockstein Separation Property, "μ-stable" for all $\aleph_0 < \mu < \kappa$ and $\{C \in [B]^{<\kappa} : C \leq B \text{ and } C \text{ is openly generated}\}$ is cofinal in $[B]^{<\kappa}$*
- *for any sufficiently large regular θ and for any $M \prec \mathcal{H}(\theta)$ s.t. $B \in M$, $\omega_1 \subseteq M$ and $|M| < \kappa$, we have that $B \cap M \leq_\sigma B$ and $B \cap M$ is openly generated.*

Corollary 10

For a regular cardinal κ, a κ-openly generated Boolean algebra B of cardinality κ has a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ consisting of openly generated Boolean algebras s.t. $B_\alpha \leq_\sigma B$ for all $\alpha < \kappa$.

Theorem 6 holds already under FRP:

Theorem 11 (F.[5])

Under FRP, the following are equivalent for any Boolean algebra B:

(a) B is openly generated.

(b) B is $\aleph_2$-openly generated.

Sketch of the Proof. (a) $\Rightarrow$ (b): By Theorem 8.

(b) $\Rightarrow$ (a): We prove the implication by induction on $|B|$.
If $|B| \leq \aleph_1$, the implication is trivial. Suppose that we have proved (b) $\Rightarrow$ (a) for all Boolean algebras of cardinality $< \kappa$. Suppose that B is $\aleph_2$-openly generated and $|B| = \kappa$.
Case I. κ is regular. In this case, B is κ-openly generated by the induction hypothesis. By Corollary 10, there is a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ of B s.t. each B_α is openly generated and $B_\alpha \leq_\sigma B$.

Theorem 6 holds already under FRP:

Theorem 11 (F.[5])

Under FRP, the following are equivalent for any Boolean algebra B:

(a) B is openly generated.

(b) B is $\aleph_2$-openly generated.

Sketch of the Proof. (a) $\Rightarrow$ (b): By Theorem 8.

(b) $\Rightarrow$ (a): We prove the implication by induction on $|B|$.
If $|B| \leq \aleph_1$, the implication is trivial. Suppose that we have proved (b) $\Rightarrow$ (a) for all Boolean algebras of cardinality $< \kappa$.
Suppose that B is $\aleph_2$-openly generated and $|B| = \kappa$.
Case I. κ is regular. In this case, B is κ-openly generated by the induction hypothesis. By Corollary 10, there is a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ of B s.t. each B_α is openly generated and $B_\alpha \leq_\sigma B$.

Theorem 6 holds already under FRP:

Theorem 11 (F.[5])

Under FRP, the following are equivalent for any Boolean algebra B:

(a) B is openly generated.

(b) B is $\aleph_2$-openly generated.

Sketch of the Proof. (a) $\Rightarrow$ (b): By Theorem 8.

(b) $\Rightarrow$ (a): We prove the implication by induction on $|B|$.
If $|B| \leq \aleph_1$, the implication is trivial. Suppose that we have proved (b) $\Rightarrow$ (a) for all Boolean algebras of cardinality $< \kappa$.
Suppose that B is $\aleph_2$-openly generated and $|B| = \kappa$.
Case I. κ is regular. In this case, B is κ-openly generated by the induction hypothesis. By Corollary 10, there is a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ of B s.t. each B_α is openly generated and $B_\alpha \leq_\sigma B$.

Theorem 6 holds already under FRP:

Theorem 11 (F.[5])

Under FRP, the following are equivalent for any Boolean algebra B:

(a) B is openly generated.

(b) B is $\aleph_2$-openly generated.

Sketch of the Proof. (a) $\Rightarrow$ (b): By Theorem 8.

(b) $\Rightarrow$ (a): We prove the implication by induction on $|B|$.
If $|B| \leq \aleph_1$, the implication is trivial. Suppose that we have proved (b) $\Rightarrow$ (a) for all Boolean algebras of cardinality $< \kappa$. Suppose that B is $\aleph_2$-openly generated and $|B| = \kappa$.
Case I. κ is regular. In this case, B is κ-openly generated by the induction hypothesis. By Corollary 10, there is a filtration $\langle B_\alpha : \alpha < \kappa\rangle$ of B s.t. each B_α is openly generated and $B_\alpha \leq_\sigma B$.

Theorem 6 holds already under FRP:

Theorem 11 (F.[5])

Under FRP, the following are equivalent for any Boolean algebra B:

(a) B is openly generated.

(b) B is $\aleph_2$-openly generated.

Sketch of the Proof. (a) $\Rightarrow$ (b): By Theorem 8.

(b) $\Rightarrow$ (a): We prove the implication by induction on $|B|$.
If $|B| \leq \aleph_1$, the implication is trivial. Suppose that we have proved (b) $\Rightarrow$ (a) for all Boolean algebras of cardinality $< \kappa$.
Suppose that B is $\aleph_2$-openly generated and $|B| = \kappa$.
Case I. κ is regular. In this case, B is κ-openly generated by the induction hypothesis. By Corollary 10, there is a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ of B s.t. each B_α is openly generated and $B_\alpha \leq_\sigma B$.

Openly generated Boolean algebras under FRP

Theorem 6 holds already under FRP:

Theorem 11 (F.[5])

Under FRP, the following are equivalent for any Boolean algebra B:

(a) B is openly generated.

(b) B is $\aleph_2$-openly generated.

Sketch of the Proof. (a) $\Rightarrow$ (b): By Theorem 8.

(b) $\Rightarrow$ (a): We prove the implication by induction on $|B|$.
If $|B| \leq \aleph_1$, the implication is trivial. Suppose that we have proved (b) $\Rightarrow$ (a) for all Boolean algebras of cardinality $< \kappa$.
Suppose that B is $\aleph_2$-openly generated and $|B| = \kappa$.
Case I. κ is regular. In this case, B is κ-openly generated by the induction hypothesis. By Corollary 10, there is a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ of B s.t. each B_α is openly generated and $B_\alpha \leq_\sigma B$.

Theorem 6 holds already under FRP:

Theorem 11 (F.[5])

Under FRP, the following are equivalent for any Boolean algebra B:

(a) B is openly generated.

(b) B is $\aleph_2$-openly generated.

Sketch of the Proof. (a) $\Rightarrow$ (b): By Theorem 8.

(b) $\Rightarrow$ (a): We prove the implication by induction on $|B|$.
If $|B| \leq \aleph_1$, the implication is trivial. Suppose that we have proved (b) $\Rightarrow$ (a) for all Boolean algebras of cardinality $< \kappa$.
Suppose that B is $\aleph_2$-openly generated and $|B| = \kappa$.
Case I. κ is regular. In this case, B is κ-openly generated by the induction hypothesis. By Corollary 10, there is a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ of B s.t. each B_α is openly generated and $B_\alpha \leq_\sigma B$.

Openly generated Boolean algebras under FRP

Theorem 6 holds already under FRP:

Theorem 11 (F.[5])

Under FRP, the following are equivalent for any Boolean algebra B:

(a) B is openly generated.

(b) B is $\aleph_2$-openly generated.

Sketch of the Proof. (a) $\Rightarrow$ (b): By Theorem 8.

(b) $\Rightarrow$ (a): We prove the implication by induction on $|B|$.
If $|B| \leq \aleph_1$, the implication is trivial. Suppose that we have proved (b) $\Rightarrow$ (a) for all Boolean algebras of cardinality $< \kappa$.
Suppose that B is $\aleph_2$-openly generated and $|B| = \kappa$.
Case I. κ is regular. In this case, B is κ-openly generated by the induction hypothesis. By Corollary 10, there is a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ of B s.t. each B_α is openly generated and $B_\alpha \leq_\sigma B$.

Suppose, toward a contradiction, that B were not openly generated. Then, by Theorem 8, $S = \{\alpha \in E^{\kappa}_{\omega} : B_\alpha \leq_{\neg\mathrm{rc}} B\}$ is stationary.

We may assume $B = \kappa$. By thinning out S, we may also assume that $B_\alpha = \alpha$ for all $\alpha \in S$. For $\alpha \in S$, there is $b_\alpha \in B$ s.t. $B_\alpha \restriction b_\alpha$ is not generated by a single element. Let $\{b_{\alpha,n} : n \in \omega\}$ be a generator of this ideal and $g(\alpha) = \{b_\alpha, n : n \in \omega\} \cup \{b_\alpha\}$.

Let $\mathcal{C} \subseteq [B]^{\aleph_1}$ be a closed unbounded set consisting of openly generated subalgebras of B. Let $I \in \mathcal{C}$ be as in the definition of FRP. Then I is openly generated.

On the other hand $b_{\alpha,n}$, $n \in \omega$, b_α for $\alpha \in S \cap I$ witness that, for any filtration $\langle I_\xi : \xi < \omega_1 \rangle$ of I, $I_\xi \leq_{\neg\mathrm{rc}} I$ for stationarily many $\xi < \omega_1$. It follows that I is not openly generated. This is a contradiction.

Suppose, toward a contradiction, that B were not openly generated. Then, by Theorem 8, $S = \{\alpha \in E^\kappa_\omega : B_\alpha \leq_{\neg\mathrm{rc}} B\}$ is stationary.

We may assume $B = \kappa$. By thinning out S, we may also assume that $B_\alpha = \alpha$ for all $\alpha \in S$. For $\alpha \in S$, there is $b_\alpha \in B$ s.t. $B_\alpha \restriction b_\alpha$ is not generated by a single element. Let $\{b_{\alpha,n} : n \in \omega\}$ be a generator of this ideal and $g(\alpha) = \{b_\alpha, n : n \in \omega\} \cup \{b_\alpha\}$.

Let $\mathcal{C} \subseteq [B]^{\aleph_1}$ be a closed unbounded set consisting of openly generated subalgebras of B. Let $I \in \mathcal{C}$ be as in the definition of FRP. Then I is openly generated.

On the other hand $b_{\alpha,n}$, $n \in \omega$, b_α for $\alpha \in S \cap I$ witness that, for any filtration $\langle I_\xi : \xi < \omega_1\rangle$ of I, $I_\xi \leq_{\neg\mathrm{rc}} I$ for stationarily many $\xi < \omega_1$. It follows that I is not openly generated. This is a contradiction.

Suppose, toward a contradiction, that B were not openly generated. Then, by Theorem 8, $S = \{\alpha \in E^\kappa_\omega : B_\alpha \leq_{\neg\mathrm{rc}} B\}$ is stationary.

We may assume $B = \kappa$. By thinning out S, we may also assume that $B_\alpha = \alpha$ for all $\alpha \in S$. For $\alpha \in S$, there is $b_\alpha \in B$ s.t. $B_\alpha \restriction b_\alpha$ is not generated by a single element. Let $\{b_{\alpha,n} : n \in \omega\}$ be a generator of this ideal and $g(\alpha) = \{b_\alpha, n : n \in \omega\} \cup \{b_\alpha\}$.

Let $\mathcal{C} \subseteq [B]^{\aleph_1}$ be a closed unbounded set consisting of openly generated subalgebras of B. Let $I \in \mathcal{C}$ be as in the definition of FRP. Then I is openly generated.

On the other hand $b_{\alpha,n}$, $n \in \omega$, b_α for $\alpha \in S \cap I$ witness that, for any filtration $\langle I_\xi : \xi < \omega_1\rangle$ of I, $I_\xi \leq_{\neg\mathrm{rc}} I$ for stationarily many $\xi < \omega_1$. It follows that I is not openly generated. This is a contradiction.

Suppose, toward a contradiction, that B were not openly generated. Then, by Theorem 8, $S = \{\alpha \in E^{\kappa}_{\omega} : B_\alpha \leq_{\neg\mathrm{rc}} B\}$ is stationary.

We may assume $B = \kappa$. By thinning out S, we may also assume that $B_\alpha = \alpha$ for all $\alpha \in S$. For $\alpha \in S$, there is $b_\alpha \in B$ s.t. $B_\alpha \restriction b_\alpha$ is not generated by a single element. Let $\{b_{\alpha,n} : n \in \omega\}$ be a generator of this ideal and $g(\alpha) = \{b_\alpha, n : n \in \omega\} \cup \{b_\alpha\}$.

Let $\mathcal{C} \subseteq [B]^{\aleph_1}$ be a closed unbounded set consisting of openly generated subalgebras of B. Let $I \in \mathcal{C}$ be as in the definition of FRP. Then I is openly generated.

On the other hand $b_{\alpha,n}$, $n \in \omega$, b_α for $\alpha \in S \cap I$ witness that, for any filtration $\langle I_\xi : \xi < \omega_1 \rangle$ of I, $I_\xi \leq_{\neg\mathrm{rc}} I$ for stationarily many $\xi < \omega_1$. It follows that I is not openly generated. This is a contradiction.

Suppose, toward a contradiction, that B were not openly generated. Then, by Theorem 8, $S = \{\alpha \in E^{\kappa}_{\omega} : B_\alpha \leq_{\neg\mathrm{rc}} B\}$ is stationary.

We may assume $B = \kappa$. By thinning out S, we may also assume that $B_\alpha = \alpha$ for all $\alpha \in S$. For $\alpha \in S$, there is $b_\alpha \in B$ s.t. $B_\alpha \restriction b_\alpha$ is not generated by a single element. Let $\{b_{\alpha,n} : n \in \omega\}$ be a generator of this ideal and $g(\alpha) = \{b_\alpha, n : n \in \omega\} \cup \{b_\alpha\}$.

Let $\mathcal{C} \subseteq [B]^{\aleph_1}$ be a closed unbounded set consisting of openly generated subalgebras of B. Let $I \in \mathcal{C}$ be as in the definition of FRP. Then I is openly generated.

On the other hand $b_{\alpha,n}$, $n \in \omega$, b_α for $\alpha \in S \cap I$ witness that, for any filtration $\langle I_\xi : \xi < \omega_1 \rangle$ of I, $I_\xi \leq_{\neg\mathrm{rc}} I$ for stationarily many $\xi < \omega_1$. It follows that I is not openly generated. This is a contradiction.

Openly generated Boolean algebras under FRP

Case II. κ is singular.

Let $\lambda = \mathrm{cf}(\kappa) < \kappa$. By the induction hypothesis, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ s.t. each B_α is openly generated.

For $\alpha < \lambda$, let $g_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ be a FN-mapping. Let $\langle C_\xi : \xi < \lambda \rangle$ be another filtration of B s.t. each C_ξ is openly generated (use the induction hypothesis) and closed w.r.t. all g_α, $\alpha < \lambda$.

Subcase IIa: $\lambda = \omega$. Then $C_\xi \leq_\sigma B$ for all $\xi < \omega$. By a theorem of Ingo Bandlow, it follows that B is openly generated.

Subcase IIb: $\lambda > \omega$. Since B satisfies the c.c.c., it follows that $C_\xi \leq_{\mathrm{rc}} B$ for all $\xi < \lambda$. By Theorem 8, it follows that B is openly generated. □

Case II. κ is singular.

Let $\lambda = \mathrm{cf}(\kappa) < \kappa$. By the induction hypothesis, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ s.t. each B_α is openly generated.

For $\alpha < \lambda$, let $g_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ be a FN-mapping. Let $\langle C_\xi : \xi < \lambda \rangle$ be another filtration of B s.t. each C_ξ is openly generated (use the induction hypothesis) and closed w.r.t. all g_α, $\alpha < \lambda$.

Subcase IIa: $\lambda = \omega$. Then $C_\xi \leq_\sigma B$ for all $\xi < \omega$. By a theorem of Ingo Bandlow, it follows that B is openly generated.

Subcase IIb: $\lambda > \omega$. Since B satisfies the c.c.c., it follows that $C_\xi \leq_{\mathrm{rc}} B$ for all $\xi < \lambda$. By Theorem 8, it follows that B is openly generated. □

Case II. κ is singular.

Let $\lambda = \mathrm{cf}(\kappa) < \kappa$. By the induction hypothesis, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ s.t. each B_α is openly generated.

For $\alpha < \lambda$, let $g_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ be a FN-mapping. Let $\langle C_\xi : \xi < \lambda \rangle$ be another filtration of B s.t. each C_ξ is openly generated (use the induction hypothesis) and closed w.r.t. all g_α, $\alpha < \lambda$.

Subcase IIa: $\lambda = \omega$. Then $C_\xi \leq_\sigma B$ for all $\xi < \omega$. By a theorem of Ingo Bandlow, it follows that B is openly generated.

Subcase IIb: $\lambda > \omega$. Since B satisfies the c.c.c., it follows that $C_\xi \leq_{\mathrm{rc}} B$ for all $\xi < \lambda$. By Theorem 8, it follows that B is openly generated. □

Case II. κ is singular.

Let $\lambda = \mathrm{cf}(\kappa) < \kappa$. By the induction hypothesis, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ s.t. each B_α is openly generated.

For $\alpha < \lambda$, let $g_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ be a FN-mapping. Let $\langle C_\xi : \xi < \lambda \rangle$ be another filtration of B s.t. each C_ξ is openly generated (use the induction hypothesis) and closed w.r.t. all g_α, $\alpha < \lambda$.

Subcase IIa: $\lambda = \omega$. Then $C_\xi \leq_\sigma B$ for all $\xi < \omega$. By a theorem of Ingo Bandlow, it follows that B is openly generated.

Subcase IIb: $\lambda > \omega$. Since B satisfies the c.c.c., it follows that $C_\xi \leq_{\mathrm{rc}} B$ for all $\xi < \lambda$. By Theorem 8, it follows that B is openly generated. □

Case II. κ is singular.

Let $\lambda = \mathrm{cf}(\kappa) < \kappa$. By the induction hypothesis, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ s.t. each B_α is openly generated.

For $\alpha < \lambda$, let $g_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ be a FN-mapping. Let $\langle C_\xi : \xi < \lambda \rangle$ be another filtration of B s.t. each C_ξ is openly generated (use the induction hypothesis) and closed w.r.t. all g_α, $\alpha < \lambda$.

Subcase IIa: $\lambda = \omega$. Then $C_\xi \leq_\sigma B$ for all $\xi < \omega$. By a theorem of Ingo Bandlow, it follows that B is openly generated.

Subcase IIb: $\lambda > \omega$. Since B satisfies the c.c.c., it follows that $C_\xi \leq_{\mathrm{rc}} B$ for all $\xi < \lambda$. By Theorem 8, it follows that B is openly generated. □

Case II. κ is singular.

Let $\lambda = \mathrm{cf}(\kappa) < \kappa$. By the induction hypothesis, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ s.t. each B_α is openly generated.

For $\alpha < \lambda$, let $g_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ be a FN-mapping. Let $\langle C_\xi : \xi < \lambda \rangle$ be another filtration of B s.t. each C_ξ is openly generated (use the induction hypothesis) and closed w.r.t. all g_α, $\alpha < \lambda$.

Subcase IIa: $\lambda = \omega$. Then $C_\xi \leq_\sigma B$ for all $\xi < \omega$. By a theorem of Ingo Bandlow, it follows that B is openly generated.

Subcase IIb: $\lambda > \omega$. Since B satisfies the c.c.c., it follows that $C_\xi \leq_{\mathrm{rc}} B$ for all $\xi < \lambda$. By Theorem 8, it follows that B is openly generated. □

Openly generated Boolean algebras under FRP

Case II. κ is singular.

Let $\lambda = \mathrm{cf}(\kappa) < \kappa$. By the induction hypothesis, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ s.t. each B_α is openly generated.

For $\alpha < \lambda$, let $g_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ be a FN-mapping. Let $\langle C_\xi : \xi < \lambda \rangle$ be another filtration of B s.t. each C_ξ is openly generated (use the induction hypothesis) and closed w.r.t. all g_α, $\alpha < \lambda$.

Subcase IIa: $\lambda = \omega$. Then $C_\xi \leq_\sigma B$ for all $\xi < \omega$. By a theorem of Ingo Bandlow, it follows that B is openly generated.

Subcase IIb: $\lambda > \omega$. Since B satisfies the c.c.c., it follows that $C_\xi \leq_{\mathrm{rc}} B$ for all $\xi < \lambda$. By Theorem 8, it follows that B is openly generated. □

Case II. κ is singular.

Let $\lambda = \mathrm{cf}(\kappa) < \kappa$. By the induction hypothesis, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ s.t. each B_α is openly generated.

For $\alpha < \lambda$, let $g_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ be a FN-mapping. Let $\langle C_\xi : \xi < \lambda \rangle$ be another filtration of B s.t. each C_ξ is openly generated (use the induction hypothesis) and closed w.r.t. all g_α, $\alpha < \lambda$.

Subcase IIa: $\lambda = \omega$. Then $C_\xi \leq_\sigma B$ for all $\xi < \omega$. By a theorem of Ingo Bandlow, it follows that B is openly generated.

Subcase IIb: $\lambda > \omega$. Since B satisfies the c.c.c., it follows that $C_\xi \leq_{\mathrm{rc}} B$ for all $\xi < \lambda$. By Theorem 8, it follows that B is openly generated. □

Case II. κ is singular.

Let $\lambda = \mathrm{cf}(\kappa) < \kappa$. By the induction hypothesis, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ s.t. each B_α is openly generated.

For $\alpha < \lambda$, let $g_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ be a FN-mapping. Let $\langle C_\xi : \xi < \lambda \rangle$ be another filtration of B s.t. each C_ξ is openly generated (use the induction hypothesis) and closed w.r.t. all g_α, $\alpha < \lambda$.

Subcase IIa: $\lambda = \omega$. Then $C_\xi \leq_\sigma B$ for all $\xi < \omega$. By a theorem of Ingo Bandlow, it follows that B is openly generated.

Subcase IIb: $\lambda > \omega$. Since B satisfies the c.c.c., it follows that $C_\xi \leq_{\mathrm{rc}} B$ for all $\xi < \lambda$. By Theorem 8, it follows that B is openly generated. □

Openly generated Boolean algebras under FRP

Case II. κ is singular.

Let $\lambda = \mathrm{cf}(\kappa) < \kappa$. By the induction hypothesis, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ s.t. each B_α is openly generated.

For $\alpha < \lambda$, let $g_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ be a FN-mapping. Let $\langle C_\xi : \xi < \lambda \rangle$ be another filtration of B s.t. each C_ξ is openly generated (use the induction hypothesis) and closed w.r.t. all g_α, $\alpha < \lambda$.

Subcase IIa: $\lambda = \omega$. Then $C_\xi \leq_\sigma B$ for all $\xi < \omega$. By a theorem of Ingo Bandlow, it follows that B is openly generated.

Subcase IIb: $\lambda > \omega$. Since B satisfies the c.c.c., it follows that $C_\xi \leq_{\mathrm{rc}} B$ for all $\xi < \lambda$. By Theorem 8, it follows that B is openly generated. $\square$

$\mathcal{L}_{\infty,\aleph_2}$-sentence describing openly generatedness

> "*B has the c.c.c., satisfies the Bockstein Separation Property and $\aleph_1$-stable and $\{C \in [B]^{\aleph_1} : C \leq B$ and C is openly generated$\}$ is cofinal in $[B]^{\aleph_1}$*"

can be formulated in an $\mathcal{L}_{\infty,\aleph_2}$-sentence. Thus, by Theorem 9, it follows that

Theorem 12

Assume FRP*. Then there is an $\mathcal{L}_{\infty,\aleph_2}$-sentence σ s.t. $B \models \sigma \Leftrightarrow B$ is openly generated.* □

$\mathcal{L}_{\infty,\aleph_2}$-sentence describing openly generatedness

"*B has the c.c.c., satisfies the Bockstein Separation Property and $\aleph_1$-stable and $\{C \in [B]^{\aleph_1} : C \leq B$ and C is openly generated$\}$ is cofinal in $[B]^{\aleph_1}$*"

can be formulated in an $\mathcal{L}_{\infty,\aleph_2}$-sentence. Thus, by Theorem 9, it follows that

Theorem 12

Assume FRP. *Then there is an $\mathcal{L}_{\infty,\aleph_2}$-sentence σ s.t. $B \models \sigma \Leftrightarrow B$ is openly generated.* □

$\mathcal{L}_{\infty,\aleph_2}$-sentence describing openly generatedness

> "*B has the c.c.c., satisfies the Bockstein Separation Property and $\aleph_1$-stable and $\{C \in [B]^{\aleph_1} : C \leq B$ and C is openly generated$\}$ is cofinal in $[B]^{\aleph_1}$*"

can be formulated in an $\mathcal{L}_{\infty,\aleph_2}$-sentence. Thus, by Theorem 9, it follows that

Theorem 12

Assume FRP*. Then there is an $\mathcal{L}_{\infty,\aleph_2}$-sentence σ s.t. $B \models \sigma \Leftrightarrow B$ is openly generated.* □

Open Problem

Is the characterization of openly generated Boolean algebras (Theorem 11) equivalent to FRP?

Open Problem

Is the characterization of openly generated Boolean algebras (Theorem 11) equivalent to FRP?

ON THE LENGTH OF PROOFS IN A FORMAL SYSTEM OF RECURSIVE ARITHMETIC

BY

TOHRU MIYATAKE

§0. Introduction.

In [6], R. J. Parikh proved the following result:

Theorem. (Parikh [6; Theorem 3]). For any formula $A(a)$, the formula $xA(x)$ is provable in PA* if and only if there is a natural number k such that all the instances $A(\bar{n})$ $(n \in \omega)$ of $A(a)$ are provable in PA* within k steps of inferences.

, where PA* is a system for Peano arithmetic formalized with only one function symbol for successor and which represents addition