Openly generated Boolean algebras under FRP

Sakaé Fuchino (渕野 昌)

Kobe University

fuchino@diamond.kobe-u.ac.jp

http://kurt.scitec.kobe-u.ac.jp/~fuchino/

(November 28, 2009 (22:47 JST) version)

神戸大学大学院 工学研究科

（旧）新井プロジェクト・プレゼンテーション室

「モデル理論とその周辺」研究集会

November 28, 2009

This presentation is typeset by pLATEX + beamer class.

Related Papers and Preprints (in chronological order)

[1] S. Fuchino, I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, *Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness*, to appear in Topology and Its Applications.

[2] S. Fuchino, *Left-separated topological spaces under Fodor-type Reflection Principle*, RIMS Kôkyûroku No.1619, (2008), 32–42.

[3] S. Fuchino, *Fodor-type Reflection Principle implies Balogh's theorems under Axiom R*, preprint.

[4] S. Fuchino H. Sakai, L. Soukup and T. Usuba, *More about the Fodor-type Reflection Principle*, in preparation.

[5] S. Fuchino, *Openly generated Boolean algebras and the Fodor-type Reflection Principle*, dedicated to Professor Dr. Sabine Koppelberg on the occasion of her retirement, preprint.

RP: For cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq [\lambda]^{\aleph_0}$, there is an $I \in [\lambda]^{\aleph_1}$ s.t. $\omega_1 \subseteq I$, $cf(I) = \omega_1$ and $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Axiom R: For cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq [\lambda]^{\aleph_0}$ and ω_1-club $\mathcal{T} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{T}$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Here, $\mathcal{T} \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-club (or "tight and unbounded" in Fleissner's terminology) if

- $\mathcal{T}$ is cofinal in $[X]^{\aleph_1}$ w.r.t. $\subseteq$ and
- for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{T}$ of length ω_1, we have $\bigcup_{\alpha<\omega_1} I_\alpha \in T$.

MM $\Rightarrow$ MA^+(σ-closed) $\Rightarrow$ Axiom R $\Rightarrow$ RP

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- ...

Fleissner's Theorem on left-separated spaces

Theorem 1 (W. Fleissner 1986)

Assume Axiom R. Suppose that X is a T_1-space with a point countable base. If X is not left-separated then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not left-separated.

- A topological space X is left-separated if there is a well-ordering $<$ of X s.t. every initial segment w.r.t. $<$ is a closed subset of X.

Fleissner's Theorem on coloring number of graphs

Theorem 2 (W. Fleissner 1986)

Assume Axiom R. If a graph (V, E) *has coloring number* $\geq \aleph_1$ *then there is an infinite subgraph of* (V, E) *of cardinality* $\aleph_1$ *with coloring number* $\aleph_1$*.*

- For a graph (V, E) the coloring number of (V, E) is the minimal cardinal μ s.t.

  there is a well-ordering $\prec$ of V s.t., for every $v \in V$, the set $\{u \in V \,:\, u \prec v,\ \{u, v\} \in E\}$ has cardinality $< \mu$.

Balogh's reflection theorem on metrizability

Theorem 3 (Z. Balogh 2002)

Assume Axiom R. Suppose that X is locally countably compact. If X is not metrizable then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not metrizable.

- A topological space X is countably compact if any countable open cover of X has a finite subcover.

- A topological space X is locally countably compact if any point of X has a neighborhood which is countably compact.

Fodor-type reflection principle (FRP)

The 'mathematical' reflection theorems mentioned above are actually consequences of the following combinatorial principle which is much weaker than Axiom R (even weaker than RP):

FRP: For any regular cardinal $\lambda \geq \aleph_2$ and stationary $S \subseteq E^\lambda_\omega$ and mapping $g : S \to [\lambda]^{\leq\aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ s.t.

- $\mathrm{cf}(I) = \omega_1$;
- $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- for any regressive $f : S \cap I \to \lambda$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \lambda$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

Some Results from F., Juhász, Szentmiklóssy and Usuba [1] (and [4])

- RP implies FRP.
- FRP is preserved by c.c.c. extension (this is not the case for RP).

$$\mathrm{MA}^{+}(\sigma\text{-closed}) \;\Rightarrow\; \text{Axiom R} \;\Rightarrow\; \mathrm{RP} \;\underset{\Rightarrow}{\not\Leftarrow}\; \mathrm{FRP} \;\underset{\Rightarrow}{\not\Leftarrow}\; \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- Fleissner's Theorem on coloring number of graphs follows from FRP
- The following reflection theorem follows from FRP: For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf
- Balogh's reflection theorem on metrizability follows from FRP

"Mathematical" characterizations of FRP

In F., Sakai, Soukup and Usuba [4] it is proved that theorems mentioned on the last slide are all equivalent to FRP over ZFC.

Corollary 4

FRP *is equivalent to each of the following assertions over* ZFC*:*

(A) *For every locally separable countably tight topological space X, if X is not meta-Lindelöf, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf.*

(B) *For every locally countably compact topological space X, if X is not metrizable, then there is a subspaces of X of cardinality $\leq \aleph_1$ which is not metrizable.*

(C) *For every T_1-space X with a point countable base, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(C') *For every metrizable space X, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(D) *For any graph $G = \langle G, \mathcal{E} \rangle$ if the coloring number of G is uncountable, then there is $I \in [G]^{\aleph_1}$ s.t. the coloring number of $G \restriction I$ is uncountable.*

Seeming extension of FRP

Lemma 5 (FRP)

Suppose that $\lambda > \aleph_1$ is a regular cardinal. Then, for any mapping $g : S \to [\lambda]^{\leq\aleph_0}$ on a stationary $S \subseteq E^\lambda_\omega$ and closed unbounded $\mathcal{C} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{C}$ as in the definition of FRP.

Proof. Suppose that λ, g, S, $\mathcal{C}$ are as above. Let sk be the canonical Skolem-hull operator on $\langle \mathcal{H}(\theta), \in, g, \mathcal{C}, \ldots, \sqsubseteq \rangle$ for a sufficiently large regular θ, where $\sqsubseteq$ is a well-ordering on $\mathcal{H}(\theta)$.
Let $C^* = \{\alpha < \kappa \,:\, sk(\alpha) \cap \kappa = \alpha\}$ and let $h : \kappa \to \kappa$ be defined by $h(\alpha) = \min C^* \setminus \alpha$ for $\alpha \in \kappa$.
Let $g' : S \to [\kappa]^{\leq\aleph_0}$ be defined by $g'(\alpha) = g(\alpha) \cup \{h(\sup g(\alpha))\}$ for $\alpha \in S$.
By FRP there is $I_0 \in [\kappa]^{\aleph_1}$ as in the definition of FRP for S and g'. Then, since C^* is closed, we have $\sup(I_0) \in C^*$. Let $I \in [\kappa]^{\aleph_1}$ be s.t. $I_0 \subseteq I \subseteq \sup(I_0)$ and $sk(I) \cap \kappa = I$. Then $I \in \mathcal{C}$ since $I = \bigcup(\mathcal{C} \cap sk(I))$ and I is as desired. □

Openly generated Boolean algebras

Notations. For Boolean algebras A, B:

- $A \leq B$: A is a subalgebra of B
- $A \restriction b = \{a \in A : a \leq b\}$ where $A \leq B$ and $b \in B$
- $A \leq_{\rm rc} B$ (A is a relatively complete subalg. of B): $A \leq B$ and $A \restriction b$ is generated by a single (largest) element for all $b \in B$
- $A \leq_{\sigma} B$ (A is a σ-subalg. of B): $A \leq B$ and $A \restriction b$ is generated by countably many elements of $A \restriction b$ for all $b \in B$
- B is openly generated $\Leftrightarrow$ $\{A \in [B]^{\aleph_0} : A \leq_{\rm rc} B\}$ contains a club subset of $[B]^{\aleph_0}$
- B is κ-openly generated $\Leftrightarrow$ $\{A \in [B]^{<\kappa} : A \leq B \text{ and } A \text{ is openly generated}\}$ contains a club subset of $[B]^{<\kappa}$.

Theorem 6 (F., Qi Feng 1993)

Under Axiom R, any Boolean algebra B is openly generated if and only if B is $\aleph_2$-openly generated.

By the following Theorem, the assertion of Theorem 6 on the last slide is independent:

Theorem 7 (F. 1993)

For a regular cardinal κ, if there is a non-reflecting stationary set $E \subseteq E^{\kappa}_{\omega}$, there is a Boolean algebra B with a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ s.t. B_α is a free Boolean algebra for all $\alpha < \kappa$ and $\{\alpha < \kappa : B_\alpha \not\leq_{\mathrm{rc}} B\} = E$.

B in the theorem above is κ-openly generated (it is even κ-free and $\mathcal{L}_{\infty,\kappa}$-free).

But B is not openly generated (by the theorem on the next slide).

Characterizations of openly generated Boolean algebras

Theorem 8

For a Boolean algebra B, the following are equivalent:

- B *is openly generated.*
- (L. Heindorf and L.B. Shapiro 1994) *There is a FN-mapping* $f : B \to [B]^{<\aleph_0}$ *(i.e. such f that, for any $a, b \in B$ with $a \leq b$ there is $c \in f(a) \cap f(b)$ s.t. $a \leq c \leq b$).*
- (F. 1994) *For any sufficiently large regular cardinal θ, and any (countable) $M \prec \mathcal{H}(\theta)$ with $B \in M$, we have $B \cap M \leq_{\mathrm{rc}} B$.*
- (F. 1994) *For any σ-closed poset $\mathbb{P}$ forcing $|B| \leq \aleph_1$, we have* $\Vdash_{\mathbb{P}}$ "B *is projective Boolean algebra*".

Theorem 9 (F. 1995)

For a Boolean algebra B and regular $\kappa > \aleph_1$, the following are equivalent:

- B *is* κ*-openly generated*
- B *has the c.c.c., satisfies the Bockstein Separation Property, "*μ*-stable" for all* $\aleph_0 < \mu < \kappa$ *and* $\{C \in [B]^{<\kappa} : C \leq B \text{ and } C \text{ is openly generated}\}$ *is cofinal in* $[B]^{<\kappa}$
- *for any sufficiently large regular* θ *and for any* $M \prec \mathcal{H}(\theta)$ *s.t.* $B \in M$, $\omega_1 \subseteq M$ *and* $|M| < \kappa$, *we have that* $B \cap M \leq_\sigma B$ *and* $B \cap M$ *is openly generated.*

Corollary 10

For a regular cardinal κ, *a* κ*-openly generated Boolean algebra* B *of cardinality* κ *has a filtration* $\langle B_\alpha : \alpha < \kappa \rangle$ *consisting of openly generated Boolean algebras s.t.* $B_\alpha \leq_\sigma B$ *for all* $\alpha < \kappa$.

Theorem 6 holds already under FRP:

Theorem 11 (F.[5])

Under FRP, the following are equivalent for any Boolean algebra B:

(a) B is openly generated.

(b) B is $\aleph_2$-openly generated.

Sketch of the Proof. (a) $\Rightarrow$ (b): By Theorem 8.

(b) $\Rightarrow$ (a): We prove the implication by induction on $|B|$.
If $|B| \leq \aleph_1$, the implication is trivial. Suppose that we have proved (b) $\Rightarrow$ (a) for all Boolean algebras of cardinality $< \kappa$. Suppose that B is $\aleph_2$-openly generated and $|B| = \kappa$.
Case I. κ is regular. In this case, B is κ-openly generated by the induction hypothesis. By Corollary 10, there is a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ of B s.t. each B_α is openly generated and $B_\alpha \leq_\sigma B$.

Suppose, toward a contradiction, that B were not openly generated. Then, by Theorem 8, $S = \{\alpha \in E^\kappa_\omega : B_\alpha \leq_{\neg\mathrm{rc}} B\}$ is stationary.

We may assume $B = \kappa$. By thinning out S, we may also assume that $B_\alpha = \alpha$ for all $\alpha \in S$. For $\alpha \in S$, there is $b_\alpha \in B$ s.t. $B_\alpha \restriction b_\alpha$ is not generated by a single element. Let $\{b_{\alpha,n} : n \in \omega\}$ be a generator of this ideal and $g(\alpha) = \{b_\alpha, n : n \in \omega\} \cup \{b_\alpha\}$.

Let $\mathcal{C} \subseteq [B]^{\aleph_1}$ be a closed unbounded set consisting of openly generated subalgebras of B. Let $I \in \mathcal{C}$ be as in the definition of FRP. Then I is openly generated.

On the other hand $b_{\alpha,n}$, $n \in \omega$, b_α for $\alpha \in S \cap I$ witness that, for any filtration $\langle I_\xi : \xi < \omega_1 \rangle$ of I, $I_\xi \leq_{\neg\mathrm{rc}} I$ for stationarily many $\xi < \omega_1$. It follows that I is not openly generated. This is a contradiction.

Case II. κ is singular.

Let $\lambda = \mathrm{cf}(\kappa) < \kappa$. By the induction hypothesis, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ s.t. each B_α is openly generated.

For $\alpha < \lambda$, let $g_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ be a FN-mapping. Let $\langle C_\xi : \xi < \lambda \rangle$ be another filtration of B s.t. each C_ξ is openly generated (use the induction hypothesis) and closed w.r.t. all g_α, $\alpha < \lambda$.

Subcase IIa: $\lambda = \omega$. Then $C_\xi \leq_\sigma B$ for all $\xi < \omega$. By a theorem of Ingo Bandlow, it follows that B is openly generated.

Subcase IIb: $\lambda > \omega$. Since B satisfies the c.c.c., it follows that $C_\xi \leq_{\mathrm{rc}} B$ for all $\xi < \lambda$. By Theorem 8, it follows that B is openly generated. □

$\mathcal{L}_{\infty,\aleph_2}$-sentence describing openly generatedness

> *"B has the c.c.c., satisfies the Bockstein Separation Property and $\aleph_1$-stable and $\{C \in [B]^{\aleph_1} : C \leq B \text{ and } C \text{ is openly generated}\}$ is cofinal in $[B]^{\aleph_1}$"*

can be formulated in an $\mathcal{L}_{\infty,\aleph_2}$-sentence. Thus, by Theorem 9, it follows that

Theorem 12

Assume FRP*. Then there is an $\mathcal{L}_{\infty,\aleph_2}$-sentence σ s.t. $B \models \sigma \Leftrightarrow B$ is openly generated.* □

Open Problem

Is the characterization of openly generated Boolean algebras (Theorem 11) equivalent to FRP?

宮武徹さんへの追悼の意を表します.

ON THE LENGTH OF PROOFS IN A FORMAL SYSTEM OF RECURSIVE ARITHMETIC

BY

TOHRU MIYATAKE

§0. Introduction.

In [6], R. J. Parikh proved the following result:

Theorem. (Parikh [6; Theorem 3]). For any formula A(a), the formula xA(x) is provable in PA* if and only if there is a natural number k such that all the instances $A(\bar{n})$ $(n \in \omega)$ of A(a) are provable in PA* within k steps of inferences.

, where PA* is a system for Peano arithmetic formalized with only one function symbol for successor and which represents addition