Pre-Hilbert spaces without orthonormal bases

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Abstract

We give an algebraic characterization of pre-Hilbert spaces with an orthonormal basis. This characterization is used to show that there are pre-Hilbert spaces X of dimension and density λ for any uncountable λ without any orthonormal basis.

Let us call a pre-Hilbert space without any orthonormal bases pathological. The pair of the cardinals $\kappa \leq \lambda$ such that there is a pre-Hilbert space of dimension κ and density λ are known to be characterized by the inequality $\lambda \leq \kappa^{\aleph_0}$. Our result implies that there are pathological pre-Hilbert spaces with dimension κ and density λ for all combinations of such κ and λ including the case $\kappa = \lambda$.

A Singular Compactness Theorem on pathology of pre-Hilbert spaces is obtained.

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A reflection theorem asserting that for any pathological pre-Hilbert space X there are stationarily many pathological sub-inner-product-spaces Y of X of smaller density is shown to be equivalent with Fodor-type Reflection Principle (FRP).

Contents

1 Introduction

An inner product space whose topology is not necessarily complete is often called a *pre-Hilbert space*.

In a pre-Hilbert space X, a maximal orthonormal system S of X does not necessarily span a dense subspace of X, that is, such S does not need to be *an orthonormal basis* (see Example 1.1 below). It is known that it is even possible that there is no orthonormal basis at all in some pre-Hilbert space (see Lemma 1.2). Let us call a pre-Hilbert space *pathological* if it does not have any orthonormal bases. If X is not pathological, i.e. if it does have an orthonormal basis, then we say that such X is *non-pathological.*

By Bessel's inequality, it is easy to see that all maximal orthonormal system S of a pre-Hilbert space X has the same cardinality independently of whether S is a basis of X or not. This cardinality is called the *dimension* of the pre-Hilbert space X and denoted by $\dim(X)$.

In the following, we fix the scalar field K of the pre-Hilbert spaces we consider in this paper to be $\mathbb{R}$ or $\mathbb{C}$ throughout.

For an infinite set S, let

(1.1) $\quad \ell_2(S) = \{\mathbf{u} \in {}^{S}K \ : \ \sum_{x\in S}(\mathbf{u}(x))^2 < \infty\},$

where $\sum_{x\in S}(\mathbf{u}(x))^2$ is defined as $\sup\{\sum_{x\in A}(\mathbf{u}(x))^2 \,:\, A \in [S]^{<\aleph_0}\}$. $\ell_2(S)$ is endowed with a natural structure of inner product space with coordinatewise addition and scalar multiplication, as well as the inner product defined by

(1.2) $\quad (\mathbf{u}, \mathbf{v}) = \sum_{x\in S} \mathbf{u}(x)\overline{\mathbf{v}(x)}$ for $\mathbf{u}$, $\mathbf{v} \in \ell_2(S)$.

It is easy to see that $\ell_2(S)$ is a/the Hilbert space of density $|\,S\,|$.

Note that any pre-Hilbert space X of density λ can be embedded densely into $\ell_2(\lambda)$ as a sub-inner-product-space. Here we call a subspace Y of a (pre-)Hilbert space X a sub-inner-product-space of X if Y is a linear subspace of X with the inner product which is the restriction of the inner product of X to Y.

For a pre-Hilbert space X and $S \subseteq X$, we denote by $[S]_X$ the sub-inner-product-space of X whose underlying set is the linear subspace of X spanned by S.

If U is a subset of $\ell_2(S)$, we denote with $\mathrm{cls}_{\ell_2(S)}(U)$ the topological closure of $[U]_{\ell_2(S)}$ in $\ell_2(S)$. We write simply $\mathrm{cls}(U)$ if it is clear in which $\ell_2(S)$ we are working.

For $x \in S$, let $\mathbf{e}^S_x \in \ell_2(S)$ be the standard unit vector at x defined by

(1.3) $\quad \mathbf{e}^S_x(y) = \delta_{x,y}$ for $y \in S$.

For $\mathbf{a} \in \ell_2(S)$, the support of $\mathbf{a}$ is defined by

(1.4) $\quad \mathrm{supp}(\mathbf{a}) = \{x \in S \,:\, \mathbf{a}(x) \neq 0\}$ $(= \{x \in S \,:\, (\mathbf{a}, \mathbf{e}^S_x) \neq 0\})$.

By the definition of $\ell^2(S)$, $\mathrm{supp}(\mathbf{a})$ is a countable subset of S for all $\mathbf{a} \in \ell^2(S)$.

For a subset U of $\ell_2(S)$ the support of U is the set $\mathrm{supp}(U) = \bigcup\{\mathrm{supp}(\mathbf{a}) \,:\, \mathbf{a} \in U\}$.

For $X \subseteq \ell_2(S)$ and $S' \subseteq S$, let $X \downarrow S' = \{\mathbf{u} \in X \,:\, \mathrm{supp}(\mathbf{u}) \subseteq S'\}$. For $\mathbf{u} \in \ell_2(S)$, let $\mathbf{u} \downarrow S' \in \ell_2(S)$ be defined by

(1.5) $\quad (\mathbf{u} \downarrow S')\,(x) = \begin{cases} \mathbf{u}(x) & \text{if } x \in S' \\ 0 & \text{otherwise} \end{cases}$

for $x \in S$. Note that $X \downarrow S'$ is not necessarily equal to $\{\mathbf{u} \downarrow S' \,:\, \mathbf{u} \in X\}$ (e.g., we have $X \downarrow \omega \neq \{\mathbf{u} \downarrow \omega \,:\, \mathbf{u} \in X\}$ where X is the pre-Hilbert space defined in Example 1.1 below).

Example 1.1 *Let X be the sub-inner-product-space of $\ell_2(\omega+1)$ spanned by $\{\mathbf{e}^{\omega+1}_n \,:\, n \in \omega\} \cup \{\mathbf{b}\}$ where $\mathbf{b} \in \ell_2(\omega+1)$ is defined by*

(1.6) $\quad \mathbf{b}(\omega) = 1$;

(1.7) $\mathbf{b}(n) = \frac{1}{n+2}$ *for* $n \in \omega$.

Then $\{\mathbf{e}^{\omega+1}_n : n \in \omega\}$ *is a maximal orthonormal system in* X *but it is not a basis of* X*.*

Proof. If $\{\mathbf{e}^{\omega+1}_n : n \in \omega\}$ were not maximal, then there would be an element $\mathbf{c}$ of X represented as a linear combination of $\mathbf{b}$ and some of $\mathbf{e}^{\omega+1}_n$'s ($n \in \omega$) such that $\mathbf{c}$ is orthogonal to all $\mathbf{e}^{\omega+1}_n$, $n \in \omega$. However, any of such linear combinations has an infinite support and hence is not orthogonal to $\mathbf{e}^{\omega+1}_n$ for any n in the support.

$\{\mathbf{e}^{\omega+1}_n : n \in \omega\}$ is not an orthonormal basis of X since $\mathrm{cls}_{\ell_2(\omega+1)}(\{\mathbf{e}^{\omega+1}_n : n \in \omega\}) = \ell_2(\omega+1) \downarrow \omega \neq \ell_2(\omega+1)$. ❑ (Example 1.1)

For all separable pre-Hilbert spaces (including the X in Example 1.1), we can always find an orthonormal basis: suppose that X is separable and let $\{\mathbf{a}_n : n \in \omega\}$ be dense in X. Then, by Gram-Schmidt orthonormalization process, we can find an orthonormal system $\{\mathbf{b}_n : n \in \omega\}$ which spans the same dense sub-inner-product-space as that spanned by $\{\mathbf{a}_n : n \in \omega\}$. Thus there are no separable pathological pre-Hilbert spaces.

The situation is different if we consider non-separable pre-Hilbert spaces.

Lemma 1.2 (P. Halmos, see Gudder [9]) *There are pre-Hilbert spaces* X *of dimension* $\aleph_0$ *and density* λ *for any* $\aleph_0 < \lambda \leq 2^{\aleph_0}$.

Note that a pre-Hilbert space X with $\dim(X) < d(X)$ cannot have any orthonormal basis, that is, such a pre-Hilbert space is pathological.

For any two pre-Hilbert spaces X, Y, the *orthogonal direct sum of* X *and* Y is the direct sum $X \oplus Y = \{\langle \mathbf{x}, \mathbf{y} \rangle : \mathbf{x} \in X, \mathbf{y} \in Y\}$ of X and Y as linear spaces together with the inner product defined by $(\langle \mathbf{x}_0, \mathbf{y}_0 \rangle, \langle \mathbf{x}_1, \mathbf{y}_1 \rangle) = (\mathbf{x}_0, \mathbf{x}_1) + (\mathbf{y}_0, \mathbf{y}_1)$ for $\mathbf{x}_0$, $\mathbf{x}_1 \in X$ and $\mathbf{y}_0$, $\mathbf{y}_1 \in Y$. A sub-inner-product-space X_0 of a pre-Hilbert space X is an *orthogonal direct summand of* X if there is a sub-inner-product-space X_1 of X such that the mapping $\varphi : X_0 \oplus X_1 \to X$; $\langle \mathbf{x}_0, \mathbf{x}_1 \rangle \mapsto \mathbf{x}_0 + \mathbf{x}_1$ is an isomorphism of pre-Hilbert spaces. If this holds, we usually identify $X_0 \oplus X_1$ with X by φ as above.

Proof of Lemma 1.2. Let B be a linear basis (Hamel basis) of the linear space $\ell_2(\omega)$ extending $\{\mathbf{e}^{\omega}_n : n \in \omega\}$. Note that $|\,B\,| = 2^{\aleph_0}$ (Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of ω of cardinality $2^{\aleph_0}$. For each $a \in \mathcal{A}$ let $\mathbf{b}_a \in \ell_2(\omega)$ be such that $\mathrm{supp}(\mathbf{b}_a) = a$. Then $\{\mathbf{b}_a : a \in \mathcal{A}\}$ is a linearly independent subset of $\ell_2(\omega)$ of cardinality $2^{\aleph_0}$).

Let $f : B \rightarrow \{\mathbf{e}_\alpha^\lambda : \alpha < \lambda\} \cup \{\mathbf{0}_{\ell_2(\lambda)}\}$ be a surjection such that $f(\mathbf{e}_n^\omega) = \mathbf{0}_{\ell_2(\lambda)}$ for all $n \in \omega$. Note that f generates a linear mapping from the linear space $\ell_2(\omega)$ to a dense subspace of $\ell_2(\lambda)$.

Let $U = \{\langle \mathbf{b}, f(\mathbf{b})\rangle : \mathbf{b} \in B\}$ and $X = [U]_{\ell_2(\omega)\oplus\ell_2(\lambda)}$. Then this X is as desired since $\{\langle \mathbf{e}_n^\omega, \mathbf{0}\rangle : n \in \omega\}$ is a maximal orthonormal system in X while we have $\mathrm{cls}_{\ell_2(\omega)\oplus\ell_2(\lambda)}(X) = \ell_2(\omega) \oplus \ell_2(\lambda)$ and hence $d(X) = \lambda$. ❏ (Lemma 1.2)

For sub-inner-product-spaces X_0, X_1 of a pre-Hilbert space X, we have $[X_0 \cup X_1]_X \cong X_0 \oplus X_1$ with the isomorphism extending

(1.8) $\quad i_{X_0\cup X_1} = \{\langle \mathbf{x}_0, \langle \mathbf{x}_0, \mathbf{0}\rangle\rangle : \mathbf{x} \in X_0\} \cup \{\langle \mathbf{x}_1, \langle \mathbf{0}, \mathbf{x}_1\rangle\rangle : \mathbf{x}_1 \in X_1\}$,

if we have

(1.9) $\quad (\mathbf{x}_0, \mathbf{x}_1) = 0$ for any $\mathbf{x}_0 \in X_0$ and $\mathbf{x}_1 \in X_1$.

Sub-inner-product-spaces X_0 and X_1 of a pre-Hilbert space X with (1.9) are said to be *orthogonal* to each other and this is denoted by $X_0 \perp X_1$.

If X_0 and X_1 are sub-inner-product-spaces of X and $X_0 \perp X_1$, we identify $[X_0 \cup X_1]_X$ with $X_0 \oplus X_1$ by the isomorphism extending the $i_{X_0\cup X_1}$ as above and write $[X_0 \cup X_1]_X = X_0 \oplus X_1$.

Similarly, if X_i, $i \in I$ are sub-inner-product-spaces of X we denote $\oplus_{i\in I} X_i = [\bigcup_{i\in I} X_i]_X$ if X_i, $i \in I$ are pairwise orthogonal, that is, if we have $X_i \perp X_j$ for all distinct i, $j \in I$.

For pairwise orthogonal sub-inner-product-paces X_i, $i \in I$ of X, we denote with $\overline{\oplus}_{i\in I}^X X_i$ the maximal linear subspace X' of X such that X' contains $\oplus_{i\in I} X_i$ as a dense subset of X'. Thus, we have $X = \overline{\oplus}_{i\in I}^X X_i$ if $\oplus_{i\in I} X_i$ is dense in X. If it is clear in which X we are working we drop the superscript X and simply write $\overline{\oplus}_{i\in I} X_i$.

An easy but very important fact for us is that

(1.10) if X_i, $i \in I$ are all non-pathological with orthonormal bases B_i for X_i, $i \in I$ and $X = \overline{\oplus}_{i\in I} X_i$, then X is also non-pathological with the orthonormal basis $\bigcup_{i\in I} B_i$.

In the following we show that there are also pathological pre-Hilbert spaces X with $\dim(X) = d(X) = \lambda$ for an uncountable λ. For regular λ this is shown in Theorem 2.1 and the general case in Corollary 5.2.

In Section 3 we prove an algebraic characterization of pre-Hilbert spaces with orthonormal bases.

In Section 4, we give a proof of the theorem by Buhagiara, Chetcutib and Weber asserting that $\kappa \leq \lambda$ are dimension and density of a pre-Hilbert space if and only if $\lambda \leq \kappa^{\aleph_0}$ holds (see Theorem 4.3). Corollary 5.2 implies that there are pathological pre-Hilbert spaces with $\dim(X) = \kappa$ and $d(X) = \lambda$ for all such κ and λ.

In sections 6, 7, 8 we study the set-theoretic reflection of the pathology of pre-Hilbert spaces.

Our set-theoretic notation is quite standard. For the basic notions and notation in set-theory we do not explain here, the reader may consult Jech [11] or Kunen [13].

2 Pathological pre-Hilbert spaces constructed from a pre-ladder system

For a cardinals λ, κ, let

(2.1) $\quad E^{\kappa}_{\lambda} = \{\alpha < \lambda \,:\, cf(\alpha) = \kappa\}$.

For $E \subseteq E^{\omega}_{\lambda}$, $\mathcal{A} = \langle A_\alpha \,:\, \alpha \in E\rangle$ is said to be a *ladder system* on E if

(2.2) $\quad A_\alpha \subseteq \alpha$ for all $\alpha \in E$;

(2.3) $\quad A_\alpha$ is cofinal in α for all $\alpha \in E$; and

(2.4) $\quad \mathrm{otp}(A_\alpha) = \omega$ for all $\alpha \in E$.

Note that, for any ladder system $\langle A_\alpha \,:\, \alpha \in E\rangle$, the sequence $\langle A_\alpha \,:\, \alpha \in E\rangle$ is pairwise almost disjoint. We shall call a sequence $\langle A_\alpha \,:\, \alpha \in E\rangle$ of countable subsets of λ a *pre-ladder system* if (2.2) holds and such that it is pairwise almost disjoint.

Theorem 2.1 *Suppose that κ is a regular cardinal $> \omega_1$, $E \subseteq E^{\omega}_{\kappa}$ is stationary and $\langle A_\xi \,:\, \xi \in E\rangle$ is a pre-ladder system such that*

(2.5) $\quad$ *$A_\xi \subseteq \xi$ consists of successor ordinals for all $\xi \in E$.*

If $\langle \mathbf{u}_\xi \,:\, \xi < \kappa\rangle$ is a sequence of elements of $\ell_2(\kappa)$ such that

(2.6) $\quad$ *$\mathbf{u}_\xi = \mathbf{e}^{\kappa}_{\xi}$ for all $\xi \in \kappa \setminus E$,*

(2.7) $\quad$ *$\mathrm{supp}(\mathbf{u}_\xi) = A_\xi \cup \{\xi\}$ for all $\xi \in E$.*

Then, letting $U = \{\mathbf{u}_\xi \,:\, \xi < \kappa\}$, $X = [U]_{\ell_2(\kappa)}$ is a pathological pre-Hilbert space of dimension and density κ.

Proof. We have $d(X) = \kappa$ since $\mathrm{cls}(X) = \ell_2(\kappa)$. $\dim(X) \leq \dim(\ell_2(\kappa)) = \kappa$ since X is a sub-inner-product-space of $\ell_2(\kappa)$ and $\dim(X) \geq \kappa$ since $\{\mathbf{u}_\alpha : \alpha \in \kappa \setminus E\}$ is an orthonormal system $\subseteq X$ of cardinality κ.

Furthermore we can show by induction that $\mathrm{cls}_{\ell_2(\kappa)}(X) \ni \mathbf{u}_\xi$ for all $\xi < \kappa$ and hence

(2.8) $\quad \mathrm{cls}_{\ell_2(\kappa)}(X) = \ell_2(\kappa)$.

To show that X is pathological, suppose toward a contradiction that $B = \langle \mathbf{b}_\xi : \xi < \kappa\rangle$ is an orthonormal basis of X.

Let χ be a sufficiently large regular cardinal and let $\langle M_\alpha : \alpha < \kappa\rangle$ be a continuously increasing sequence of elementary submodels of $\mathcal{H}(\chi)$ such that

(2.9) $\quad |\, M_\alpha \,| < \kappa$ for all $\alpha < \kappa$,

(2.10) $\quad \langle A_\xi : \xi \in E\rangle, \langle \mathbf{u}_\xi : \xi < \kappa\rangle, \langle \mathbf{b}_\xi : \xi < \kappa\rangle \in M_0$,

(2.11) $\quad \kappa_\alpha = \kappa \cap M_\alpha \in \kappa$ for all $\alpha < \kappa$ and $\langle \kappa_\alpha : \alpha < \kappa\rangle$ is a strictly increasing sequence of ordinals cofinal in κ.

For $\alpha < \kappa$, let $H_\alpha = \ell_2(\kappa) \downarrow \kappa_\alpha$. Note that H_α is a closed sub-inner-product-space of $\ell_2(\kappa)$ isomorphic to $\ell_2(\kappa_\alpha)$.

Let $B_\alpha = \{\mathbf{b}_\xi : \xi < \kappa_\alpha\}$ for $\alpha < \kappa$.

Claim 2.1.1 $\mathrm{supp}(B_\alpha) \subseteq \kappa_\alpha$ *and* B_α *is an orthonormal basis of* H_α*.*

$\vdash$ For $\xi < \kappa_\alpha$ $\mathbf{b}_\xi \in M_\alpha$ by (2.10). Hence $\mathrm{supp}(\mathbf{b}_\xi) \in M_\alpha$. Since $\mathrm{supp}(\mathbf{b}_\xi)$ is countable it follows that $\mathrm{supp}(\mathbf{b}_\xi) \subseteq \kappa \cap M_\alpha = \kappa_\alpha$. Thus we have $\mathrm{supp}(B_\alpha) \subseteq \kappa_\alpha$.

For $\eta < \kappa_\alpha$, we have

(2.12) $\quad \mathcal{H}(\chi) \models$ "there are $A \in [\kappa]^{\aleph_0}$ and $c \in {}^A K$ such that $\sum_{\xi\in A} c(\xi)\mathbf{b}_\xi = \mathbf{u}_\eta$"

since $\langle \mathbf{b}_\xi : \xi < \kappa\rangle$ is an orthonormal basis. By (2.10) and elementarity, it follows that

(2.13) $\quad M_\alpha \models$ "there are $A \in [\kappa]^{\aleph_0}$ and $c \in {}^A K$ such that $\sum_{\xi\in A} c(\xi)\mathbf{b}_\xi = \mathbf{u}_\eta$".

Let $A \in [\kappa]^{\aleph_0} \cap M_\alpha$ and $c \in {}^A\kappa \cap M_\alpha$ be witnesses of (2.13). Since A is countable we have $A \subseteq M_\alpha$. Thus $\mathbf{u}_\eta$ is a limit of linear combinations of elements of B_α.

It follows that $\mathrm{cls}\left([B_\alpha]_{H_\alpha}\right) \supseteq \mathrm{cls}(\{\mathbf{u}_\xi : \xi < \kappa_\alpha\}) = H_\alpha$. $\quad\dashv$ (Claim 2.1.1)

Since E is stationary, there is an $\alpha^* < \kappa$ such that $\kappa_{\alpha^*} \in E$. Let $\kappa^* = \kappa_{\alpha^*}$.

Claim 2.1.2 *For any nonzero* $\mathbf{a} \in X$ *represented as a linear combination of finitely many elements of* U *including (a non-zero multiple of)* $\mathbf{u}_{\kappa*}$*, we have* $\mathbf{a} \notin H_{\kappa^*}$ *and there is* $\xi < \kappa^*$ *such that* $(\mathbf{a}, \mathbf{b}_\xi) \neq 0$*.*

$\vdash$ Suppose that

(2.14) $\mathbf{a} = c\mathbf{u}_{\kappa*} + \sum_{\xi\in s} a_\xi \mathbf{u}_\xi + \sum_{\eta\in t} b_\eta \mathbf{u}_\eta$

where $s \in [\kappa^*]^{<\aleph_0}$, $t \in [\kappa \setminus (\kappa^* + 1)]^{<\aleph_0}$ and c, a_ξ, $b_\eta \in K \setminus \{0\}$ for $\xi \in s$ and $\eta \in t$.

$S = \bigcup\{\text{supp}(\mathbf{u}_\xi \,:\, \xi \in s \cup t\}$ is a union of finite subset of $\kappa \setminus \{\kappa^*\}$ and finitely many A_ξ, $\xi \in SE \setminus \{\kappa^*\}$. It follows that S is almost disjoint to A_κ and it does not contain κ^*. In particular $\text{supp}(\mathbf{a}) \cap \kappa^*$ is non-empty and $\mathbf{a} \notin H_{\kappa^*}$.

Thus $\mathbf{a} \downarrow \kappa^*$ is a non-zero element of H_{α^*}. By Claim 2.1.1, it follows that there is $\xi < \kappa^*$ such that $(\mathbf{a}, \mathbf{b}_\xi) = (\mathbf{a} \downarrow \kappa^*, \mathbf{b}_\xi) \neq 0$. $\dashv$ (Claim 2.1.2)

By Claim 2.1.2, there are no $\mathbf{a} \in X$ as in the assertion of Claim 2.1.2 among $\mathbf{b}_\xi$, $\xi < \kappa$. It follows that $\kappa^* \notin \bigcup\{\text{supp}(\mathbf{b}_\xi) \,:\, \xi < \kappa\}$. It follows that $\mathbf{e}^\kappa_{\kappa^*} \notin \text{cls}_{\ell_2(\kappa)}(B)$. By (2.8), this is a contradiction to the assumption that $\{\mathbf{b}_\xi \,:\, \xi < \kappa\}$ is an orthonormal basis of X. ❑ (Theorem 2.1)

The construction of X in Theorem 2.1 can be further modified to obtain the following additional property of X: there is $\mathcal{S} \subseteq [U]^{<\kappa}$ such that

(2.15) $\mathcal{S}$ is a stationary subset of $[U]^{<\kappa}$,

(2.16) for all A, $B \in \mathcal{S}$ with $A \subseteq B$, $[A]_X$ is an orthogonal direct summand of $[B]_X$.

For (2.15) and (2.15), we can just start from a stationary and co-stationary E and let

(2.17) $\mathcal{S} = \{U_\gamma \,:\, \gamma \in \kappa \setminus E\}$

where $U_\gamma = \{\mathbf{u}_\xi \,:\, \xi < \gamma\}$. Then U and this $\mathcal{S}$ are as desired: $\mathcal{S}$ is a stationary subset of $[U]^{<\kappa}$ by the choice of E. For U_{γ_0}, $U_{\gamma_1} \in \mathcal{S}$ with $\gamma_0 < \gamma_1$, we have $\mathbf{u}_\xi \downarrow (\kappa \setminus \gamma_0) \in [U_{\gamma_1}]_X$ for all $\xi \in \gamma_1 \setminus \gamma_0$. Hence

(2.18) $[U_{\gamma_1}]_X = [U_{\gamma_0}]_X \oplus [\{\mathbf{u}_\xi \downarrow (\kappa \setminus \gamma_0) \,:\, \xi \in \gamma_1 \setminus \gamma_0\}]_X$.

Theorem 2.1 applied to $\kappa = \omega_1$ gives pathological pre-Hilbert spaces with interesting properties. Note that for a stationary subset E of ω_1 there is a partial ordering which "shoots" a club subset inside E while preserving all cardinals (e.g. the shooting a club forcing with finite conditions).

If X is a pre-Hilbert space constructed as in Theorem 2.1 for stationary and co-stationary $E \subseteq E^{\omega}_{\omega_1}$ and a pre-ladder system on E, letting $U \subseteq \ell_2(\omega_1)$ be the generator of X as in Theorem 2.1, we have that $X \downarrow \alpha$ is non-pathological for all $\alpha < \omega_1$ since $X \downarrow \alpha$ is separable. If we shoot a club subset of $\omega_1 \setminus E$, we obtain a continuously increasing sequence of non-pathological sub-inner-product-spaces $\langle X_\alpha : \alpha < \omega_1 \rangle$ of X such that $\bigcup_{\alpha<\omega_1} X_\alpha = X$ and that X_α is an orthogonal direct summand of $X_{\alpha+1}$ for all $\alpha < \omega_1$. It follows that X is non pathological in such a generic extension. Thus we obtain:

Corollary 2.2 (1) *There is a pathological pre-Hilbert space X of dimension and density $\aleph_1$ such that there is a partial ordering $\mathbb{P}$ preserving all cardinals such that $\Vdash_{\mathbb{P}}$ "X has an orthonormal basis".*

(2) *There is a pathological pre-Hilbert space X of dimension and density $\aleph_1$ such that, for any partial ordering $\mathbb{P}$ preserving ω_1, we have $\Vdash_{\mathbb{P}}$ "X is pathological".*

Proof. A proof of (1) is already explained above. For (2), we can use the club set $E = E^{\omega}_{\omega_1}$ in the construction of the proof of Theorem 2.1. The pre-Hilbert space X constructed in the proof of Theorem 2.1 with this E is as desired: since E^* remains stationary in any generic extension preserving ω_1, X remains pathological there.

❑ (Corollary 2.2)

3 A Characterization of the non-pathology

Using some of the ideas in the proof of Theorem 2.1, we obtain an "algebraic" characterization of pre-Hilbert spaces with orthonormal bases (see Theorem 3.3). This characterization is used in later sections.

Lemma 3.1 *Suppose that X is a pre-Hilbert space and X is a dense sub-inner-product-space of $\ell_2(S)$. If $\mathcal{B} \subseteq X$ is an orthonormal basis then, for any $S_0 \subseteq S$, there is an $A \subseteq S$ such that $S_0 \subseteq A$, $|A| = |S_0| + \aleph_0$, $X \downarrow A$ is a dense sub-inner-product-space of $\ell_2(S) \downarrow A$, $\mathcal{B}_A = \{\mathbf{b} \in \mathcal{B} : \mathrm{supp}(\mathbf{b}) \subseteq A\}$ is an orthonormal basis of $X \downarrow A$ and $\mathcal{B}^-_A = \mathcal{B} \setminus \mathcal{B}_A$ is an orthonormal basis of $X \downarrow (S \setminus A)$. In particular, we have $X = (X \downarrow A) \oplus (X \downarrow (S \setminus A))$.*

Proof. Let χ be a sufficiently large regular cardinal and let $M \prec \mathcal{H}(\chi)$ be such that

$$K,\, X,\, S,\, \mathcal{B} \in M,\ S_0 \subseteq M \text{ and } |\, M\,| = |\, S_0\,| + \aleph_0. \tag{3.1}$$

We show that $A = S \cap M$ is as desired. Since $S_0 \subseteq M$, we have $S_0 \subseteq S \cap M = A$. Since $\mathcal{B}$ is also an orthonormal basis of $\ell_2(S)$, we have

$$\mathcal{H}(\chi) \models \text{"there is a } B \in [\mathcal{B}]^{\aleph_0} \text{ and } c \in {}^{B}K \text{ such that } \textstyle\sum_{\mathbf{u}\in B} c(\mathbf{u})\mathbf{u} = \mathbf{e}^S_s\text{"} \tag{3.2}$$

for all $s \in A$. By elementarity, it follows that

$$M \models \text{"there is a } B \in [\mathcal{B}]^{\aleph_0} \text{ and } c \in {}^{B}K \text{ such that } \textstyle\sum_{\mathbf{u}\in B} c(\mathbf{u})\mathbf{u} = \mathbf{e}^S_s\text{"}. \tag{3.3}$$

Let $B \in [\mathcal{B}]^{\aleph_0} \cap M$ and $c \in {}^{B}K \cap M$ be witnesses of (3.3). By $B \in M$ and since B is countable, we have $B \subseteq M$. For each $\mathbf{b} \in B$, since $\mathbf{b} \in M$ and $\mathrm{supp}(\mathbf{b})$ is countable, we have $\mathrm{supp}(\mathbf{b}) \subseteq M$. It follows that $B \subseteq \mathcal{B}_A$ and $\mathbf{e}^S_s \in \mathrm{cls}_{\ell_2(S)\downarrow A}(\mathcal{B}_A)$ for all $s \in A$.

Thus $\{\mathbf{e}^S_s \,:\, s \in A\} \subseteq \mathrm{cls}_{\ell_2(S)\downarrow A}(\mathcal{B}_A)$ and hence

$$\mathrm{cls}_{\ell_2(S)\downarrow A}(\mathcal{B}_A) = \ell_2(S) \downarrow A. \tag{3.4}$$

Since $\mathcal{B}_A \subseteq X \downarrow A$, (3.4) implies that $X \downarrow A$ is dense in $\ell_2(S) \downarrow A$.

For any $\mathbf{b} \in \mathcal{B}^-_A$, we have $\mathrm{supp}(\mathbf{b}) \subseteq S \setminus A$: otherwise, $\mathbf{b} \downarrow A \neq \mathbf{0}_{\ell_2(S)}$. By (3.4) it follows that there is a $\mathbf{c} \in \mathcal{B}_A$ such that $(\mathbf{b}, \mathbf{c}) = (\mathbf{b} \downarrow A, \mathbf{c}) \neq 0$. This is a contradiction to the orthonormality of $\mathcal{B}$. Thus $\mathcal{B}^-_A \subseteq X \downarrow (S \setminus A)$.

$\mathcal{B}^-_A$ is an orthonormal basis of $X \downarrow (S \setminus A)$: similarly to the argument above, it is enough to show that, for each $s \in S \setminus A$, $\mathbf{e}^S_s$ can be obtained as a (possibly infinite) sum of elements in $\mathcal{B}^-_A$ in $\ell_2(S) \downarrow (S \setminus A)$. Since $\mathcal{B}$ is an orthonormal basis of $\ell_2(S)$, we have $\mathbf{e}^S_s = \sum_{\mathbf{b}\in B}(\mathbf{e}^S_s, \mathbf{b})\mathbf{b}$ where $B = \{\mathbf{b} \in \mathcal{B} \,:\, (\mathbf{e}^S_s, \mathbf{b}) \neq 0\}$. Since $\mathrm{supp}(\mathbf{b}) \not\subseteq A$ for all $\mathbf{b} \in B$, we have $B \subseteq \mathcal{B}^-_A$. ❏ (Lemma 3.1)

Lemma 3.2 *Suppose that X is a non-pathological pre-Hilbert space and X is a dense sub-inner-product space of $\ell_2(S)$ for some infinite set S. Then there is a partition $\mathcal{P}$ of S into countable subsets such that $X = \overline{\bigoplus}_{A\in\mathcal{P}} X \downarrow A$.*

Proof. Let $|\, S\,| = \kappa$ and $\mathcal{B} = \{\mathbf{b}_\alpha \,:\, \alpha < \kappa\}$ be an orthonormal basis of X. Let $S = \{s_\alpha \,:\, \alpha < \kappa\}$.

We define by induction on $\alpha \in \kappa$ the sequences $\langle S_\alpha \,:\, \alpha < \kappa\rangle$ and $\langle A_\alpha \,:\, \alpha < \kappa\rangle$ of subsets of S such that:

$$S_0 = S; \tag{3.5}$$

(3.6) $A_\alpha \in [S_\alpha]^{\aleph_0}$ for all $\alpha \in \kappa$;

(3.7) $S_{\alpha+1} = S_\alpha \setminus A_\alpha$ for all $\alpha \in \kappa$;

(3.8) $S_\gamma = \bigcap_{\alpha<\gamma} S_\alpha$ for all limit $\gamma \in \kappa$;

(3.9) $s_\alpha \in \bigcup_{\beta\leq\alpha} A_\alpha$ for all $\alpha \in \kappa$;

(3.10) $\mathcal{B} \cap (X \downarrow S_\alpha)$ is an orthonormal basis of $X \downarrow S_\alpha$ for all $\alpha \in \kappa$; and

(3.11) $\mathcal{B} \cap (X \downarrow A_\alpha)$ is an orthonormal basis of $X \downarrow A_\alpha$ for all $\alpha \in \kappa$.

The construction of A_α and $S_{\alpha+1}$ is possible by Lemma 3.1. We just have to check that the construction of S_γ at limit steps $\gamma < \kappa$ works.

For a limit $\gamma < \kappa$ we have $S_\gamma = \bigcap_{\alpha<\gamma} S_\alpha$ by (3.8). For each $s \in S_\gamma$ and $\alpha < \gamma$ there are a countable $B_\alpha \subseteq \mathcal{B} \cap (X \downarrow S_\alpha)$ and a sequence $\langle a^\alpha_{\mathbf{b}} : \mathbf{b} \in B_\alpha \rangle$ in K such that

(3.12) $\mathbf{e}^S_s = \sum_{\mathbf{b}\in B_\alpha} a^\alpha_{\mathbf{b}} \mathbf{b}$.

By the uniqueness of the representation of elements of $\ell_2(S)$ as an infinite linear combination of elements of $\mathcal{B}$. It follows that there is a countable $B^* \subseteq \mathcal{B} \cap (X \downarrow S_\alpha)$ and a sequence $\langle a_{\mathbf{b}} : \mathbf{b} \in B^* \rangle$ such that $B_\alpha = B^*$ for all $\alpha < \gamma$ and $a^\alpha_{\mathbf{b}} = a_{\mathbf{b}}$ for all $\alpha < \gamma$ and $\mathbf{b} \in B^*$. It follows that $B^* \subseteq \mathcal{B} \cap (X \downarrow B_\gamma)$.

Thus, we have $\mathbf{e}^S_s \in \mathrm{cls}_{\ell_2(S)}[\mathcal{B} \cap (X \downarrow S_\gamma)]$, for all $s \in S_\gamma$. It follows that $\mathcal{B} \cap (X \downarrow S_\gamma)$ is an orthonormal basis of $X \downarrow S_\gamma$, i.e. S_γ satisfies (3.10).

$\mathcal{P} = \{A_\alpha : \alpha < \kappa\}$ is then a partition of S as desired. ❑ (Lemma 3.2)

Theorem 3.3 *Suppose that X is a pre-Hilbert space. Then X is non-pathological if and only if there are separable sub-inner-product-spaces X_α, $\alpha < \delta$ of X such that $X = \overline{\bigoplus}_{\alpha<\delta} X_\alpha$.*

Proof. If X is separable then the claim is trivial with $\delta = 1$.

Suppose that X is non-separable.

If X is non-pathological then there are separable sub-inner-product-spaces X_α, $\alpha < \kappa$ for $\kappa = d(X)$ with $X = \overline{\bigoplus}_{\alpha<\kappa} X_\alpha$ by Lemma 3.2.

Conversely, if there are X_α, $\alpha < \delta$ as above, then each X_α for $\alpha \in \delta$ has an orthonormal basis B_α. $\mathcal{B} = \bigcup_{\alpha<\delta} B_\alpha$ is then an orthonormal basis of X. ❑ (Theorem 3.3)

Lemma 3.4 *Suppose that X is a non-pathological pre-Hilbert space and X is a dense sub-inner-product space of $\ell_2(S)$ for some uncountable set S. Then there is a filtration $\langle S_\alpha : \alpha < \kappa \rangle$ of S for $\kappa = cf(|S|)$ such that $X \downarrow S_\alpha$ is an orthogonal direct summand of X for all $\alpha < \kappa$.*

Proof. By Lemma 3.2 there is a partition $\mathcal{P}$ of S into countable subsets such that $X = \overline{\bigoplus}_{P\in\mathcal{P}} X \downarrow P$. Let $\langle \mathcal{P}_\alpha : \alpha < \kappa\rangle$ be a filtration of $\mathcal{P}$ and let $S_\alpha = \bigcup \mathcal{P}_\alpha$ for $\alpha < \kappa$. Then $\langle S_\alpha : \alpha < \kappa\rangle$ is as desired. ❑ (Lemma 3.4)

The following Lemmas are used in Section 8. We put them together here since they stand in a similar context as that of previous results in this section.

Lemma 3.5 *Suppose that X is a pre-Hilbert-space which is a dense sub-inner-product-space of $\ell_2(S)$. For $S' \subseteq S$ such that*

(3.13) *$X \downarrow S'$ is dense in $\ell_2 \downarrow S'$,*

$X \downarrow S'$ is not an orthogonal direct summand of X if and only if there is $\mathbf{a} \in X$ such that

(3.14) *$\mathbf{a} \downarrow S' \notin X$.*

Proof. If there is no $\mathbf{a} \in X$ with (3.14) then we clearly have $X = (X \downarrow S') \otimes (X \downarrow S \backslash S')$.

Suppose that $\mathbf{a} \in X$ satisfies (3.14). Note that then we have $\mathrm{supp}(\mathbf{a}) \nsubseteq S'$ and $\mathrm{supp}(\mathbf{a}) \cap S' \neq \emptyset$. Suppose toward a contradiction that there is a sub-inner-product space X'' of X such that

(3.15) $X = (X \downarrow S') \oplus X''$.

Then there are $\mathbf{a}' \in X \downarrow S'$ and $\mathbf{a}'' \in X''$ such that $\mathbf{a} = \mathbf{a}' + \mathbf{a}''$. So $\mathbf{a}'' = \mathbf{a} - \mathbf{a}'$. It follows that $\mathbf{a}'' \downarrow S' \neq \mathbf{0}$ by (3.14). By (3.13), there is some $\mathbf{b} \in X \downarrow S'$ such that $(\mathbf{a}'', \mathbf{b}) = (\mathbf{a}'' \downarrow S', \mathbf{b}) \neq 0$. This is a contradiction to (3.15). ❑ (Lemma 3.5)

Lemma 3.6 *Suppose that X is a pre-Hilbert-space which is a dense sub-inner-product-space of $\ell_2(S)$. For a sufficiently large regular χ and $M \prec \mathcal{H}(\chi)$ with K, X, $S \in M$, $X \downarrow (S \cap M)$ is dense in $\ell_2(S) \downarrow (S \cap M)$.*

Proof. For $s \in S \cap M$, we have

(3.16) $\mathcal{H}(\chi) \models$ there are $A \in [X]^{\aleph_0}$ and $c \in {}^{A}K$ such that $\mathbf{e}^S_s = \sum_{\mathbf{b}\in A} c(\mathbf{b})\mathbf{b}$.

By elementarity it follows that

(3.17) $M \models$ there are $A \in [X]^{\aleph_0}$ and $c \in {}^{A}K$ such that $\mathbf{e}^S_s = \sum_{\mathbf{b}\in A} c(\mathbf{b})\mathbf{b}$.

Let $A \in [X]^{\aleph_0} \cap M$ and $c \in {}^{A}K \cap M$ be witnesses of (3.17). By the countability of A we have $A \subseteq M$ and, for each $\mathbf{b} \in A$, $\mathrm{supp}(\mathbf{b}) \subseteq M$ since $\mathrm{supp}(\mathbf{b})$ is countable.

This shows that $\mathbf{e}^S_s \in \mathrm{cls}(X \downarrow (S \cap M))$. ❑ (Lemma 3.6)

Lemma 3.7 *Suppose that X is a pre-Hilbert-space which is a dense sub-inner-product-space of $\ell_2(S)$ for an uncountable S. Then there is a filtration $\langle S_\alpha : \alpha < \kappa \rangle$ of S such that $X \downarrow S_\alpha$ dense in $\ell_2(S) \downarrow S_\alpha$ for all $\alpha < \kappa$*

Proof. Let χ be a sufficiently large regular cardinal. Let $\kappa = cf(|\, S \,|)$ and let $\langle M_\alpha : \alpha < \kappa \rangle$ be a continuously increasing sequence of elementary submodels of $\mathcal{H}(\chi)$ such that

(3.18) K, X, $S \in M_0$,

(3.19) $|\, M_\alpha \,| < |\, S \,|$ for all $\alpha < \kappa$,

(3.20) $S \subseteq \bigcup_{\alpha<\kappa} M_\alpha$.

Letting $S_\alpha = S \cap M_\alpha$ for $\alpha < \kappa$, the sequence $\langle S_\alpha : \alpha < \kappa \rangle$ is as desired by Lemma 3.6.
❑ (Lemma 3.7)

4 Dimension and density of pre-Hilbert spaces

The proof of Lemma 1.2 actually yields pre-Hilbert spaces of the following combinations of dimension and density:

Lemma 4.1 (A generalization of Lemma 1.2) *For any cardinal κ and λ with $\kappa < \lambda \le \kappa^{\aleph_0}$, there are (pathological) pre-Hilbert spaces of dimension κ and density λ.* ❑

On the other hand if $\kappa^{\aleph_0} < \lambda$ there are no pre-Hilbert space X with dimension κ and density λ.

Proposition 4.2 (David Buhagiara, Emmanuel Chetcutib and Hans Weber [1], see also [4]) *For any pre-Hilbert space X, we have $d(X) \le |\, X \,| \le (\dim(X))^{\aleph_0}$.*

Proof. Let X be a pre-Hilbert space. We may assume without loss of generality that X is a dense sub-inner-product-space of the Hilbert space $\ell_2(\kappa)$ for $\kappa = d(X) > \dim(X) \ge \aleph_0$.

Let $\mathcal{B} = \langle \mathbf{b}_\xi : \xi < \kappa \rangle$ be a maximal orthonormal system in X and $D = \bigcup\{\mathrm{supp}(\mathbf{b}_\xi) : \xi < \kappa\}$. By the assumption we have $|\, D \,| = \kappa$.

Claim 4.2.1 *For any distinct* $\mathbf{a}_0$, $\mathbf{a}_1 \in X$ *we have* $\mathbf{a}_0 \restriction D \neq \mathbf{a}_1 \restriction D$.

Proof. Suppose that there were $\mathbf{a}_0$, $\mathbf{a}_1 \in X$ such that $\mathbf{a}_0 \neq \mathbf{a}_1$ but $\mathbf{a}_0 \restriction D = \mathbf{a}_1 \restriction D$. Then $\mathbf{a}_2 = \mathbf{a}_1 - \mathbf{a}_0$ would be a non-zero element of X orthogonal to all $\mathbf{b}_\xi$, $\xi < \kappa$. This is a contradiction to the maximality of $\mathcal{B}$. ⊣ (Claim 4.2.1)

Let $\varphi : \ell_2(D) \to X$ be defined by

$$(4.1) \quad \varphi(\mathbf{c}) = \begin{cases} \text{the unique } \mathbf{a} \in X \text{ such that } \mathbf{c} = \mathbf{a} \restriction D; & \text{if there is such } \mathbf{a} \in X, \\ \mathbf{0}; & \text{otherwise} \end{cases}$$

for $\mathbf{c} \in \ell_2(D)$. φ is well-defined by Claim 4.2.1 and it is surjective. Thus we have

$$(4.2) \quad d(X) \leq |\, X \,| \leq |\, \ell_2(D) \,| = (\dim(X))^{\aleph_0}.$$

❑ (Proposition 4.2)

The following theorem will be yet extended in Corollary 5.4.

Theorem 4.3 *For any cardinal* $\kappa \leq \lambda$ *there is a pre-Hilbert space of dimension* κ *and density* λ *if and only if* $\lambda \leq \kappa^{\aleph_0}$ *holds.*

Proof. For $\kappa = \lambda$, $\ell_2(\kappa)$ is an example of pre-Hilbert space of dimension and density κ and λ. If $\kappa < \lambda < \kappa^{\aleph_0}$, Lemma 4.1 provides an example.

The converse also holds by Proposition 4.2. ❑ (Theorem 4.3)

5 Orthogonal direct sum

In a variety $\mathcal{V}$ of algebraic structures it can happen that there is a non free algebra $A \in \mathcal{V}$ such that the product $A \otimes F$ is free for some free algebra $F \in \mathcal{V}$. For example, it is known that there are non-free projective algebra B in the variety $\mathcal{B}$ of Boolean algebras but free product $B \oplus F$ of any projective algebra B with a sufficiently large free Boolean algebra F is free.

In contrast, the pathology of pre-Hilbert space remains by orthogonal direct sum.

Theorem 5.1 *For any pre-Hilbert spaces* X_0 *and* X_1*, the orthogonal direct sum* $X_0 \oplus X_1$ *is pathological if and only if at least one of* X_0 *and* X_1 *is pathological.*

Proof. If X_0 and X_1 are both non-pathological and $\mathcal{B}_0$ and $\mathcal{B}_1$ are orthonormal bases of X_0 and X_1 respectively, then $\mathcal{B}_0 \times \{\mathbf{0}_{X_2}\} \cup \{\mathbf{0}_{X_1}\} \times \mathcal{B}_1$ is an orthonormal basis of $X_0 \oplus X_1$.

Conversely, suppose that $X_0 \oplus X_1$ is non-pathological and $\mathcal{B}$ is an orthonormal basis of $X = X_0 \oplus X_1$. Without loss of generality, we may assume that there are S, S^0, S^1 such that $S = S^0 \cup S^1$, $S^0 \cap S^1 = \emptyset$, X_i is a dense sub-inner-product-space of $\ell_2(S) \downarrow S^i$ for $i \in 2$ and $X_0 \oplus X_1 = [X_0 \cup X_1]_{\ell_2(S)}$.

By Lemma 3.2, there is a partition $\langle A_\alpha : \alpha < \delta \rangle$ of S into countable sets such that $X = \overline{\bigoplus}_{\alpha\in\kappa} X \downarrow A_\alpha$. We may assume that the elements of partition A_α in the proof of Lemma 3.2 is obtained in the construction as the intersection of S_α (in the proof of Lemma 3.2) and countable $M_\alpha \prec \mathcal{H}(\chi)$ such that $\mathcal{B}$, X_0, X_1, S^0, $S^1, \ldots \in M_\alpha$. Then as in the proof of Lemma 3.1, we have $X \downarrow A_\alpha = (X_0 \downarrow (A_{0,\alpha})) \oplus (X_1 \downarrow (A_{1,\alpha}))$ where $A_{i,\alpha} = A_\alpha \cap S^i$ for $i \in 2$.

Let $\mathcal{P}_i = \{A_{i,\alpha} : \alpha < \kappa,\ A_{i,\alpha} \neq \emptyset\}$ for $i \in 2$. Then $X_i = \overline{\bigoplus}_{P\in\mathcal{P}_i} X_i \downarrow P$ for $i \in 2$. Thus X_i, $i \in 2$ are non-pathological. ❑ (Theorem 5.1)

Corollary 5.2 *For any uncountable cardinal λ, there is a pathological pre-Hilbert space Z of dimension and density λ.*

Proof. Let X be any pathological pre-Hilbert space with density $\aleph_1$. Then $Z = X \oplus \ell_2(\lambda)$ has dimension and density λ. Z is pathological by Theorem 5.1. ❑ (Corollary 5.2)

Corollary 5.3 *For any infinite cardinals κ and λ with $\kappa \leq \lambda \leq \kappa^{\aleph_0}$ there is a pathological pre-Hilbert space of dimension κ and density λ.*

Proof. By Lemma 4.1 and Corollary 5.2. ❑ (Corollary 5.3)

Corollary 5.4 (1) *For any infinite cardinals κ and λ with $\kappa \leq \lambda \leq \kappa^{\aleph_0}$ there is a pathological pre-Hilbert space of dimension κ and density λ such that there is a partial ordering $\mathbb{P}$ preserving all cardinals such that $\Vdash_{\mathbb{P}}$ "X is non-pathological".*

(2) *For any infinite cardinals κ and λ with $\kappa \leq \lambda \leq \kappa^{\aleph_0}$ there is a pathological pre-Hilbert space of dimension κ and density λ which remains pathological in any generic extension preserving ω_1.*

Proof. The pre-Hilbert space of the form $X \oplus Y$ will do where X is as in Corollary 2.2,(1) or (2) and Y is as in Corollary 5.3. ❑ (Corollary 5.4)

6 Reflection and non-reflection of pathology

For any pre-Hilbert space X all sub-inner-product-spaces of X of density $\aleph_0$ are non-pathological. If $S \subseteq E^{\omega}_{\omega_2}$ is non-reflecting stationary set, then the sub-inner-product-space of $\ell_2(\omega_2)$ constructed from a ladder system on S, there are club many $\beta < \omega_2$ such that $X \downarrow \beta$ is non-pathological.

A similar non-reflection theorem holds at an arbitrary regular uncountable cardinal $\kappa > \aleph_1$ under a weak form of the square principle at κ.

For a regular cardinal κ, $\mathsf{ADS}^-(\kappa)$ is the assertion that there is a stationary set $S \subseteq E^{\omega}_{\kappa}$ and a sequence $\langle A_\alpha : \alpha \in S\rangle$ such that

(6.1) $A_\alpha \subseteq \alpha$ and $\mathrm{otp}(A_\alpha) = \omega$ for all $\alpha \in S$;

(6.2) for any $\beta < \kappa$, there is a mapping $f : S \cap \beta \to \beta$ such that $f(\alpha) < \sup(A_\alpha)$ for all $\alpha \in S \cap \beta$ and $A_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ are pairwise disjoint

(for more about $\mathsf{ADS}^-(\kappa)$, see Fuchino, Juhaász, Soukup, Szentmiklóssy, Usuba [5] and Fuchino, Sakai, Soukup [7]).

We shall call $\langle A_\alpha : \alpha \in S\rangle$ as above an $\mathsf{ADS}^-(\kappa)$-sequence. Note that it follows from (6.1) and (6.2) that A_α, $\alpha \in S$ are pairwise almost disjoint.

Under $\mathsf{ADS}^-(\kappa)$, we may further assume that the $\mathsf{ADS}^-(\kappa)$-sequence $\langle A_\alpha : \alpha \in S\rangle$ satisfies that $A_\alpha \subseteq \alpha \setminus \mathrm{Lim}$ for all $\alpha \in S$.

Since an $\mathsf{ADS}^-(\kappa)$-sequence is a pre-ladder system, we can apply the construction of pre-Hilbert spaces in the proof of Theorem 2.1 to the sequence and obtain the following:

Theorem 6.1 *Assume that $\mathsf{ADS}^-(\kappa)$ holds for a regular cardinal $\kappa > \omega_1$. Then there is a pathological dense sub-inner-product-space X of $\ell_2(\kappa)$ such that $X \downarrow \beta$ is non-pathological for all $\beta < \kappa$. Furthermore for any regular $\lambda < \kappa$, $\{S \in [\kappa]^{\lambda} : X \downarrow S$ is non-pathological$\}$ contains a club subset of $[\kappa]^{\lambda}$.*

Proof. Let $\langle A_\alpha : \alpha \in E\rangle$ be an $\mathsf{ADS}^-(\kappa)$-sequence on a stationary $E \subseteq E^{\omega}_{\kappa}$. Let $\langle \mathbf{u}_\xi : \xi < \kappa\rangle$ be a sequence of elements of $\ell_2(\kappa)$ with (2.6) and (2.7), $U = \{\mathbf{u}_\xi : \xi < \kappa\}$ and $X = [U]_{\ell_2(\kappa)}$. Then X is pathological by Theorem 2.1.

For $\beta < \kappa$ let $U_\beta = \{\mathbf{u}_\xi : \xi < \beta\}$. We show that $X_\beta = [U_\beta]_{\ell_2(\kappa)}$ is non-pathological.

Let $f : E \cap \beta \to \beta$ be as in (6.2). For each $\alpha \in E \cap \beta$, let $B_\alpha = (A_\alpha \setminus f(\alpha)) \cup \{\alpha\}$. Then B_α, $\alpha \in E \cap \beta$ are pairwise disjoint. Let $C = \beta \setminus (\bigcup_{\alpha \in E \cap \beta} B_\alpha)$.

Note that

(6.3) $\mathbf{u}'_\alpha = \mathbf{u}_\alpha - \sum_{\xi\in A_\alpha\cap f(\alpha)} \mathbf{u}_\alpha(\xi)\mathbf{e}^\kappa_\xi$

is an element of X and $\mathrm{supp}(\mathbf{u}'_\alpha) = B_\alpha$. It follows that X_β is the orthogonal sum of the sub-inner-product-spaces $X \downarrow C$, $X \downarrow B_\alpha$, $\alpha \in E \cap \beta$. In particular, we have $X_\beta = \overline{\bigoplus}_{\alpha\in E\cap\beta} X \downarrow B_\alpha \ \oplus\ X \downarrow C$. Note that from this it follows that $X_\beta = X \downarrow \beta$.

Now $X \downarrow B_\alpha$, $\alpha \in E \cap \beta$ are non-pathological since they are separable. Let U_α be an orthonormal basis of $X \downarrow B_\alpha$ for $\alpha \in E \cap \beta$. Also $X \downarrow C$ is non-pathological with the orthonormal basis $\{\mathbf{e}^\kappa_\alpha : \alpha \in C\}$. Thus $\bigcup_{\alpha\in E\cap\beta} U_\alpha \cup \{\mathbf{e}^\kappa_\alpha : \alpha \in C\}$ is an orthonormal basis of X_β.

The same argument shows that $X \downarrow S$ is non-pathological for any bounded subset S of κ closed with respect to the sequence $\langle A_\alpha : \alpha \in E\rangle$ (that is, $A_\alpha \subseteq S$ for all $\alpha \in E \cap S$). Note that, for all regular $\lambda < \kappa$ there are club many such S of cardinality λ. ❑ (Theorem 6.1)

Under the consistency strength of certain very large cardinals we obtain reflection theorems for pathology of pre-Hilbert spaces.

Theorem 6.2 *Suppose that κ is a supercompact cardinal. Then for any pathological pre-Hilbert space X, there are stationarily many pathological sub-inner-product-spaces Y of X of size $< \kappa$.*

Proof. Suppose that X is a pathological pre-Hilbert space of size λ. We may assume that the underlying set of X is λ. If $\lambda < \kappa$ then the statement of the theorem is trivial. So we assume that $\lambda \geq \kappa$. Let $\mathcal{C} \subseteq [\lambda]^{<\kappa}$ be a club set. Let $j : V \overset{\preceq}{\to} M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^\lambda M \subseteq M$. Then we have $j''X \in M$ and $j''X \in j(\mathcal{C})$: the latter is because $M \models$ "$j(\mathcal{C})$ is a club subset of $[j(\lambda)]^{<j(\kappa)}$" and $\mathcal{D} = \{j(Y) : Y \in \mathcal{C}\} \subseteq j(\mathcal{C})$ is of cardinality $< j(\kappa)$ with $\bigcup\mathcal{D} = j''X$. We have $V \models j''X \cong X$ and hence $V \models$ "$j''X$ is pathological". It follows that $M \models$ "$j''X$ is pathological". Putting these facts together, we obtain

(6.4) $M \models$" $j''X$ is a sub-inner-product space of $j(X)$, $j''X \in j(\mathcal{C})$ and
$j''X$ is pathological".

Thus,

(6.5) $M \models$"there is a pathological sub-inner-product-space Y of $j(X)$ with $Y \in j(\mathcal{C})$.

By elementarity if follows

(6.6) $V \models$"there is a pathological sub-inner-product-space Y of X with $Y \in \mathcal{C}$.

❑ (Theorem 6.2)

Theorem 6.3 *Suppose that X is a pathological pre-Hilbert space and X is a dense sub-inner-product-space of $\ell_2(S)$ for some infinite set S. Then for any ccc partial ordering $\mathbb{P}$ we have $\Vdash_{\mathbb{P}}$ "$[X]_{\ell_2(S)}$ is pathological".*

Proof. Suppose that X is a pre-Hilbert space and there is a ccc partial ordering $\mathbb{P}$ such that

(6.7) $\Vdash_{\mathbb{P}}$ "$[X]_{\ell_2(S)}$ is non-pathological".

We show that X is then non-pathological.

By Theorem 3.2 and the Maximal Principle there is a $\mathbb{P}$-name $\underset{\sim}{\mathcal{P}}$ of partition of S into countable sets such that

(6.8) $\Vdash_{\mathbb{P}}$ "$[X]_{\ell_2(S)} = \overline{\bigoplus}_{P\in\underset{\sim}{\mathcal{P}}}[X]_{\ell_2(S)} \downarrow P$".

Claim 6.3.1 *There is a partition $\mathcal{P}'$ of κ into countable sets such that, for each $P \in \mathcal{P}'$, we have $\Vdash_{\mathbb{P}}$ "P is a countable union of elements of $\underset{\sim}{\mathcal{P}}$".*

$\vdash$ Let $\sim$ be the transitive closure of the relation

(6.9) $\sim_0 = \{\langle s,t\rangle \in S$: there is $p \in \mathbb{P}$ such that
$p \Vdash_{\mathbb{P}}$ "s and t belong to the same set $\in \underset{\sim}{\mathbb{P}}$".$\}$

By the ccc of $\mathbb{P}$, $Q_s = \{t \in S : s \sim_0 t\}$ is countable for all $s \in S$. Hence all equivalence classes of $\sim$ are also countable.

Let $\mathcal{P}'$ be the partition of S into equivalence classes of $\sim$.

Let $P \in \mathcal{P}'$. We show that $\Vdash_{\mathcal{P}}$ "P is a union of elements of $\underset{\sim}{\mathcal{P}}$". Let G be an arbitrary $(V,\mathbb{P})$-generic set In $V[G]$ suppose that $s \in P$, $s \in Q$ for some $Q \in \underset{\sim}{\mathcal{P}}^G$ and $t \in Q$. Then there is some $p \in G$ such that $p \Vdash_{\mathbb{P}}$ "s and t are in the same element of $\underset{\sim}{\mathcal{P}}$". It follows that $s \sim_0 t$ and $t \in P$. $\dashv$ (Claim 6.3.1)

Claim 6.3.2 *For $P \in \mathcal{P}'$ we have $X = (X \downarrow P) \oplus (X \downarrow (S \setminus P))$.*

$\vdash$ Let G be a $(V,\mathbb{P})$-generic set. In $V[G]$, we have $[X]_{\ell_2(S)} = ([X]_{\ell_2(S)} \downarrow P) \oplus ([X]_{\ell_2(S)} \downarrow (S \setminus P))$ by Claim 6.3.1. Hence, in V, we have $X = (X \downarrow P) \oplus (X \downarrow (S \setminus P))$
$\dashv$ (Claim 6.3.2)

It follows from Claim 6.3.2 that $X = \overline{\bigoplus}_{P \in \mathcal{P}'} X \downarrow P$. Thus, by Theorem 3.3, X is has an orthonormal basis. ❑ (Theorem 6.3)

The Cohen forcing $\mathrm{Fn}(\kappa, 2)$ in the following theorem can be replaced by may other c.c.c. forcing notions which can be seen as iterations with certain coherence (see Dow, Tall Weiss [2]).

Theorem 6.4 *Assume that κ is a supercompact cardinal and let $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$. Then we have*

(6.10) $\Vdash_{\mathbb{P}}$ "*for every pathological pre-Hilbert space X which is a dense sub-inner-product space of $\ell_2(\lambda)$ for some infinite λ, there are stationarily many $S \in [\lambda]^{<2^{\aleph_0}}$ such that $X \downarrow S$ is pathological*".

Proof. Let G be a $(V,\mathbb{P})$-generic filter. Working in $V[G]$, let X be a pre-Hilbert space which is a dense sub-inner-product-space of $\ell_2(\lambda)$. If $\lambda < \kappa = \left(2^{\aleph_0}\right)^{V[G]}$ then the assertion is trivial. Thus we assume $\lambda \geq \kappa$. Let $\mathcal{C} \subseteq [\lambda]^{<\kappa}$ be a club set. It is enough to show that there is some $S \in \mathcal{C}$ such that $X \downarrow S$ is pathological.

Back in V, let $j : V \overset{\preceq}{\to} M$ be a λ-supercompact embedding. That is, the elementary embedding j is such that $M \subseteq V$ is a transitive class $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$.

Let $\mathbb{P}^* = \mathrm{Fn}(j(\kappa), 2) = j(\mathbb{P})$ and let G^* be a $(V, \mathbb{P}^*)$-generic filter with $G^* \supseteq G$.

Let $j^* : V[G] \overset{\preceq}{\to} M[G^*]$ be the extension of j defined by

(6.11) $j^*([\underset{\sim}{a}]^G) = [j(\underset{\sim}{a})]^{G^*}$

for each $\mathbb{P}$-name $\underset{\sim}{a}$. It is easy to check that j^* is well-defined and

(6.12) $\left({}^{\lambda}M[G^*]\right)^{V[G^*]} \subseteq M[G^*]$.

It follows that $j^{*\prime\prime}X \in M[G^*]$ and $\mathrm{supp}(j^{*\prime\prime}X) = j^{\prime\prime}\lambda \in j^*(\mathcal{C})$. Since $V[G^*]$ is a c.c.c. extension of $V[G]$, by Lemma 6.3, we have

(6.13) $V[G^*] \models$ "$[j^{\prime\prime}X]_{\ell_2(j(\lambda))}$ is pathological".

It follows that

(6.14) $M[G^*] \models$ "$[j''X]_{\ell_2(j(\lambda))}$ is pathological"

by the same argument as right after (6.4).

Thus we have

(6.15) $M[G^*] \models$"there is $S \in j^*(\mathcal{C})$ such that $j(X) \downarrow S$ is pathological".

By elementarity it follows that

(6.16) $V[G] \models$"there is $S \in \mathcal{C}$ such that $X \downarrow S$ is pathological".

❏ (Theorem 6.4)

7 A Singular Compactness Theorem

The proof of the following theorem follows closely the proof of Shelah's Singular Compactness Theorem given in Hodges [10]. A similar Singular Compactness Theorem in the context of (non-)freeness of modules is given in Eklof [3].

Theorem 7.1 *Suppose that λ is a singular cardinal and X is a pre-Hilbert space which is a dense sub-inner-product-space of $\ell_2(\lambda)$. If X is pathological then there is a cardinal $\lambda' < \lambda$ such that*

(7.1) $\{u \in [\lambda]^{\kappa^+} : X \downarrow u$ *is a pathological pre-Hilbert space*$\}$

is stationary in $[\lambda]^{\kappa^+}$ for all $\lambda' \leq \kappa < \lambda$.

In the following we shall prove the contraposition of the statement of the theorem:

Theorem 7.1* *For any singular λ and any pre-Hilbert space X which is a dense sub-inner-product-space of $\ell_2(\lambda)$, if*

(7.2) $\mathcal{N}_\kappa^X = \{u \in [\lambda]^{\kappa^+} : X \downarrow u$ *is a non-pathological pre-Hilbert space*$\}$ *contains a club in $[\lambda]^{\kappa^+}$ for cofinally many $\kappa < \lambda$,*

then X is non-pathological.

For a dense sub-inner-product space X of $\ell_2(\lambda)$ and v, $v' \subseteq \lambda$, we write $u' \parallel_X u$ if $u \subseteq u'$, $X \downarrow u$ and $X \downarrow u'$ are dense in $\ell_2(\lambda) \downarrow u$ and $\ell_2(\lambda) \downarrow u'$ respectively; and $X \downarrow u$ is an orthogonal direct summand of $X \downarrow u'$, i.e. if $X \downarrow u' = (X \downarrow u) \oplus (X \downarrow (u' \setminus u))$, see Lemma 3.5.

For a cardinal κ, the κ-Shelah game over $X \subseteq \ell_2(\lambda)$ (notation $\mathcal{G}_\kappa(X)$) is the game whose matches $\mathcal{M}$ are ω-sequences of moves by Players I and II

$$\mathcal{M}: \begin{array}{lllll} \text{I} & u_0 & u_1 & u_2 & \cdots \\ \text{II} & v_0 & v_1 & v_2 & \cdots \end{array}$$

where u_i, $v_i \in [\lambda]^\kappa$ for $i \in \omega$ and $u_0 \subseteq v_0 \subseteq u_1 \subseteq v_1 \subseteq u_2 \subseteq v_2 \subseteq \cdots$.

Player II wins if $X \downarrow v_i$ is non-pathological and $v_{i+1} \parallel_X v_i$ for all $i \in \omega$.

Note that, if Player II wins in a match $\mathcal{M}$ with the moves $u_0 \subseteq v_0 \subseteq u_1 \subseteq v_1 \subseteq u_2 \subseteq v_2 \subseteq \cdots$, then $X \downarrow w$ for $w = \bigcup_{i\in\omega} u_i = \bigcup_{i\in\omega} v_i$ is non-pathological.

Lemma 7.2 *κ-Shelah game over $X \subseteq \ell_2(\lambda)$ is determined for regular κ.*

Proof. Since the game is open for Player I, the proof of Gale-Stewart Theorem applies (see e.g. Kanamori [12] or Hodges [10]). ❑ (Lemma 7.2)

Lemma 7.3 *Suppose that X is a dense sub-inner-product-space of $\ell_2(\lambda)$ for a cardinal λ. For a cardinal $\kappa < \lambda$, if $\mathcal{N}_\kappa^X$ contains a club subset of $[\lambda]^{\kappa^+}$, then Player II has a winning strategy in $\mathcal{G}_\kappa(X)$.*

Proof. By Lemma 7.2, it is enough to show that the Player I does not have a winning strategy.

Suppose that σ is a strategy for Player I. We show that it is not winning.

Let $\mathcal{C} \subseteq \mathcal{M}_\kappa^X$ be club in $[\lambda]^{\kappa^+}$.

Let χ be a sufficiently large regular cardinal and let $\langle M_\alpha : \alpha < \kappa^+\rangle$ be a continuously increasing chain of elementary submodels of $\mathcal{H}(\theta)$ such that

(7.3) σ, X, λ, κ, $\mathcal{C}, \ldots \in M_0$;

(7.4) $|\, M_\alpha \,| = \kappa$ and $M_\alpha \in M_\alpha + 1$ for all $\alpha < \kappa^+$;

(7.5) $\alpha \subseteq M_\alpha$ for all $\alpha < \kappa^+$

(7.6) For any finite subsequence $\mathcal{G}_0$ of $\langle M_\beta : \beta \leq \alpha\rangle$, if $\mathcal{G}_0$ is the moves of Player II in an initial segment $\mathcal{M}_0$ of a match in $\mathcal{G}_\kappa(X)$ where the Player I has played according to σ and the last member of $\mathcal{G}_0$ is the last move in $\mathcal{M}_0$, then $\sigma(\mathcal{M}_0) \in M_{\alpha+1}$ and $\sigma(\mathcal{M}_0) \subseteq M_{\alpha+1}$.

Let $M = \bigcup_{\alpha<\kappa^+} M_\alpha$. By (7.3), (7.4) and (7.5), we have

(7.7) $\lambda \cap M \in \mathcal{C}$.

By Theorem 3.3, there is a partition $\mathcal{P}$ of $\lambda \cap M$ into countable sets such that $X \downarrow (\lambda \cap M) = \overline{\bigoplus}_{A\in\mathcal{P}} X \downarrow A$. Let $C = \{\alpha < \kappa^+ : \lambda \cap M_\alpha$ is a union of some elements of $\mathcal{P}\}$. Then C is a club set $\subseteq \kappa^+$, (7.8) $M \downarrow (\lambda \cap M_\alpha)$ is non-pathological for all $\alpha \in C$ and

(7.9) $(\lambda \cap M_\alpha) \parallel_X (\lambda \cap M)$ for every $\alpha \in C$.

Let α_i, $i \in \omega$ be the first ω elements of C and $v_i = \lambda \cap M_{\alpha_i}$ for $i \in \omega$. By (7.6), there is a match $\mathcal{M}$ in $\mathcal{G}_\kappa(X)$ in which Player I has chosen his moves according to σ and $\langle v_i : i \in \omega \rangle$ is the moves of Player II. Player II wins in this match $\mathcal{M}$ by (7.9). This shows that σ is not a winning strategy of Player I. ❑ (Lemma 7.3)

Proof of Theorem 7.1*: Suppose that X and λ are as in Theorem 7.1*. Let $\delta = cf(\lambda)$ and let $\langle \lambda_\xi : \xi < \delta \rangle$ be a continuously increasing sequence of cardinals below λ such that

(7.10) $\delta < \lambda_0$;

(7.11) $\mathcal{N}^X_{\lambda_\xi}$ (defined in (7.2)) contains a club subset $\subseteq [\lambda]^{(\lambda_\xi)^+}$ for all successor $\xi < \delta$.

The condition (7.11) is possible by our assumption (7.10).

In the following, we construct u^i_ξ, $\tilde{u}^i_\xi$, v^i_ξ for $\xi < \delta$ and $i \in \omega$ such that

(7.12) $\lambda_\xi = u^0_\xi \subseteq \tilde{u}^0_\xi \subseteq v^0_\xi \subseteq u^1_\xi \subseteq \tilde{u}^1_\xi \subseteq v^1_\xi \subseteq u^2_\xi \subseteq \tilde{u}^2_\xi \subseteq v^2_\xi \subseteq \cdots$

and, letting $w_\xi = \bigcup_{i\in\omega} u^i_\xi = \bigcup_{i\in\omega} \tilde{u}^i_\xi = \bigcup_{i\in\omega} v^i_\xi$, we have

(7.13) $\langle w_\xi : \xi \in \delta \rangle$ is a filtration of λ;

(7.14) $X \downarrow w_\xi$ is non-pathological for all $\xi \in \delta$;

(7.15) $w_\eta \parallel_X w_\xi$ for all $\xi < \eta < \delta$.

From (7.13), (7.14) and (7.15), it follows immediately that X is non-pathological.

For the construction of u^i_ξ, $\tilde{u}^i_\xi$, v^i_ξ for $\xi < \delta$ and $i \in \omega$, we fix winning strategies σ_ξ for Player II in $\mathcal{G}_{\lambda_\xi}(X)$ for all successor $\xi < \delta$. We have such strategies by (7.11) and Lemma 7.3.

The following describes the inductive construction:

(7.16) $|\, u^i_\xi \,| = |\, \tilde{u}^i_\xi \,| = |\, v^i_\xi \,| = \lambda_\xi$ for all $\xi < \delta$;

(7.17) The sequence $\tilde{u}^0_\xi, v^0_\xi, \tilde{u}^1_\xi, v^1_\xi, \tilde{u}^2_\xi, v^2_\xi, \ldots$ is a match in $\mathcal{G}_{\lambda_\xi}(X)$ in which Player II has played according to σ_ξ for all successor $\xi < \delta$ ((7.14) for all successor $\xi < \delta$ follows from this);

(7.18) When $\langle u^k_\xi : k \leq i, \xi < \delta\rangle$, $\langle \tilde{u}^j_\xi : j < i, \xi < \delta\rangle$ and $\langle v^j_\xi : j < i, \xi < \delta\rangle$ have been chosen (according to all the conditions described here) for an $i \in \omega$ then $\tilde{u}^i_\xi$ for each $\xi < \delta$ is such that $\tilde{u}^i_\xi \supseteq \bigcup_{\eta\leq\xi} u^i_\eta$ holds (note that $|\bigcup_{\eta\leq\xi} u^i_\eta| = \lambda_\xi$ by (7.16). This condition guarantees that the sequence $\langle w_\xi : \xi < \delta\rangle$ is going to be increasing);

For each successor $\xi < \delta$ and $i \in \omega$, if v^i_ξ has been chosen according to the conditions described here, $X \downarrow v^i_\xi$ is non-pathological by (7.17). Thus we can find a partition $\mathcal{P}^i_\xi$ of v^i_ξ into countable sets such that $X \downarrow v^i_\xi = \overline{\oplus}_{A\in\mathcal{P}^i_\xi} X \downarrow A$ by Theorem 3.3. If $i > 0$ then we may choose $\mathcal{P}^i_\xi$ such that $\mathcal{P}^{i-1}_\xi \subseteq \mathcal{P}^i_\xi$ (this is possible since $v^i_\xi \parallel_X v^{i-1}_\xi$ by (7.17)).

(7.19) (a continuation of (7.18)) When $\langle u^k_\xi : k \leq i, \xi < \delta\rangle$, $\langle \tilde{u}^j_\xi : j < i, \xi < \delta\rangle$ and $\langle v^j_\xi : j < i, \xi < \delta\rangle$ have been chosen (according to all the conditions described here) for an $i \in \omega$ then we choose $\tilde{u}^i_\xi$ also such that $\tilde{u}^i_\xi \cap v^k_{\xi+1}$ is a union of some elements of $\mathcal{P}^k_\xi$ for all $k < i$ for all (not necessarily successor) $\xi < \delta$ (this makes $w_{\xi+1} \parallel_X w_\xi$ for all $\xi < \delta$);

For each $\xi < \delta$ and $i \in \omega$, when v^i_ξ has been chosen, we enumerate it as $v^i_\xi = \{\beta_{i,\xi,\eta} : \eta < \lambda_\xi\}$.

(7.20) When $\langle u^j_\xi : j < i, \xi < \delta\rangle$, $\langle \tilde{u}^j_\xi : j < i, \xi < \delta\rangle$ and $\langle v^j_\xi : j < i, \xi < \delta\rangle$ have been chosen (according to all the conditions described here) for an $i \in \omega$ then we let $u^{i+1}_\xi = \{\beta_{i,\xi,\eta} : \xi < \delta, \eta < \lambda_\xi\} \cup v^i_\xi$
(this is possible since the set on the right side of the inequality has size $\leq \lambda_\xi$. This condition makes the sequence $\langle w_\xi : \xi < \delta\rangle$ continuous).

To see that (7.20) makes the sequence $\langle w_\xi : \xi < \delta\rangle$ continuous, suppose that $\nu \in w_\gamma$ for a limit $\gamma < \delta$. Then there is $i^* \in \omega$ such that $\nu \in v^{i^*}_\gamma$. Hence there is $\eta^* < \lambda_\gamma$ such that $\nu = \beta_{i^*,\gamma,\eta^*}$. Let $\xi < \gamma$ be such that $\eta^* < \lambda_\xi$. Then by (7.20) we have $\nu = \beta_{i^*,\gamma,\eta^*} \in u^{i^*+1}_\xi \subseteq w_\xi$.

As noted above, the choice of u^i_ξ, $\tilde{u}^i_\xi$, v^i_ξ for $\xi < \delta$ and $i \in \omega$ with (7.12), (7.16) $\sim$ (7.20) makes $\langle w_\xi : \xi < \delta\rangle$ satisfy the conditions (7.13), (7.14) for all successor $\xi < \delta$ and (7.15) for all $\xi < \delta$ and $\eta = \xi + 1$.

By the continuity of $\langle w_\xi : \xi < \delta\rangle$ we can then prove inductively that (7.14) and (7.15) hold for all $\xi < \eta < \delta$. ❏ (Theorem 7.1*)

8 Reflection of pathology and Fodor-type Reflection Principle

In this section we prove the following theorem which gives characterizations of FRP in terms of pathology of pre-Hilbert spaces.

Theorem 8.1 *Each of the following assertions is equivalent to* FRP*:*

(8.1) *For any regular $\kappa > \omega_1$ and any dense sub-inner-product-space X of $\ell_2(\kappa)$, if X is pathological then*

$$S_X = \{\alpha < \kappa \,:\, X \downarrow \alpha \textit{ is pathological}\}$$

is stationary in κ.

(8.2) *For any regular $\kappa > \omega_1$ and any dense sub-inner-product-space X of $\ell_2(\kappa)$, if X is pathological then*

$$S_X^{\aleph_1} = \{U \in [\kappa]^{\aleph_1} \,:\, X \downarrow U \textit{ is pathological}\}$$

is stationary in $[\kappa]^{\aleph_1}$.

First let us review some facts around the reflection principle FRP needed for the proof of Theorem 8.1.

One of the combinatorial statements equivalent to FRP we are going to use below is as follows:

(FRP) For any regular $\kappa > \omega_1$, any stationary $E \subseteq E^\omega_\kappa$ and any mapping $g : E \to [\kappa]^{\aleph_0}$, there is $\alpha^* \in E^{\omega_1}_\kappa$ such that

(8.3) α^* is closed with respect to g (that is, $g(\alpha) \subseteq \alpha^*$ for all $\alpha \in E \cap \alpha^*$) and, for any $I \in [\alpha^*]^{\aleph_1}$ closed with respect to g, closed in α^* with respect to the order topology and with $\sup(I) = \alpha^*$, if $\langle I_\alpha \,:\, \alpha < \omega_1\rangle$ is a filtration of I then $\sup(I_\alpha) \in E$ and $g(\sup(I_\alpha)) \cap \sup(I_\alpha) \subseteq I_\alpha$ hold for stationarily many $\alpha < \omega_1$

(see Fuchino, Sakai, Soukup [7]).

FRP was invented by Lajos Soukup and the author in 2008 and then published in Fuchino Juhaász, Soukup, Szentmiklóssy, Usuba [5] by a formulation slightly different from the one given above. In Fuchino, Sakai, Soukup [7] it is proved that FRP is equivalent to the statement that $\mathsf{ADS}^-(\kappa)$ fails for all regular $\kappa > \omega_1$. This characterization of FRP is used to show the equivalence of FRP to many mathematical reflection statements in Fuchino [6], Fuchino, Sakai, Soukup [7], Fuchino, Rinot [8]. One of the typical mathematical assertion equivalent with FRP is:

> For every non-metrizable countably compact topological space X there is a non-metrizable subspace of X of cardinality $\leq \aleph_1$ (see [7]).

Our present result adds another couple of mathematical reflection statements to the long list of the statements equivalent to FRP.

For the proof of Theorem 8.1 we need the following easy observations:

Lemma 8.2 (cf. Lemma 6.1 in [5]) *Suppose that κ is a regular cardinal $> \aleph_1$, $C \subseteq \kappa$ club, $E \subseteq C$ stationary and $a_\eta \in [\kappa]^{\aleph_0}$ for $\eta \in E$. Then there is a stationary $E' \subseteq E^\omega_\kappa \cap C$ and a mapping $\overline{\eta} : E' \to E;\ \xi \mapsto \eta_\xi$ such that, for all $\xi \in E'$, we have $\xi \leq \eta_\xi$ and $a_{\eta_\xi} \cap \xi = a_{\eta_\xi} \cap \eta_\xi$.*

Proof. We prove the Lemma in the following two cases:
Case I. $E \cap E^\omega_\kappa$ is stationary.

Then $E' = E \cap E^\omega_\kappa$ with $\bar{\eta} = \mathrm{id}_{E'}$ is as desired.

Case II. $E \cap E^\omega_\kappa$ is non-stationary. Then $E'' = E \setminus E^\omega_\kappa$ is stationary. For each $\eta \in E''$ we have $\sup(a_\eta \cap \eta) < \eta$. By Fodor's Lemma there are $\eta_0 < \kappa$ and stationary $E''' \subseteq E''$ such that $\sup(a_\eta \cap \eta) \leq \eta_0$ for all $\eta \in E'''$.

Let $E' = (E^\omega_\kappa \cap C) \setminus \eta_0$ and, for each $\xi \in E'$, let $\eta_\xi = \min(E''' \setminus \xi)$. Then this E' with $\bar{\eta} : E' \to E;\ \xi \mapsto \eta_\xi$ is as desired. ❏ (Lemma 8.2)

Lemma 8.3 *Suppose that κ and λ are regular cardinals with $\aleph_0 < \kappa < \lambda$ and A a set of size $\geq \lambda$. If $S \subseteq [A]^{<\lambda}$ is stationary in $[A]^{<\lambda}$ and $U_s \subseteq [s]^{<\kappa}$ is stationary for all $s \in S$, then $\bigcup_{s\in S} U_s$ is stationary in $[A]^{<\kappa}$.*

Proof. Suppose that $C \subseteq [A]^{<\kappa}$ is club. Then there is $f : [A]^{<\aleph_0} \to [A]^{<\kappa}$ such that

(8.4) $\quad C_f = \{x \in [A]^{<\kappa} : x \text{ is closed with respect to } f\} \subseteq C$

(see e.g. Lemma 8.26 in Jech [11]). Note that then

(8.5) $\quad C_f^{<\lambda} = \{y \in [A]^{<\lambda} : y \text{ is closed with respect to } f\}$

is a club $\subseteq [A]^{<\lambda}$.

Since S is a stationary subset of $[A]^{<\lambda}$, there is $s^* \in S \cap C_f^{<\lambda}$. Now $C_f \cap [s^*]^{<\kappa}$ is a club in $[s^*]^{<\kappa}$ and U_{s^*} is stationary in $[s^*]^{<\kappa}$.

Thus there is $u^* \in U_{s^*} \cap (C_f \cap [s^*]^{<\kappa}) \subseteq (\bigcup_{s \in S} U_s) \cap C_f \subseteq (\bigcup_{s \in S} U_s) \cap C$.

❑ (Lemma 8.3)

Proof of Theorem 8.1: First we show that FRP implies (8.1).

Assume that FRP holds. Suppose that X is a dense sub-inner-product-space of $\ell_2(\kappa)$ for a regular cardinal $\kappa > \aleph_1$. We assume that S_X (in (8.1)) is non-stationary and drive a contradiction.

By the assumption there is a club set $C \subseteq \kappa$ such that $X \downarrow \alpha$ is non-pathological for all $\alpha \in C$. By Lemma 3.7 we may assume that $X \downarrow \alpha$ is dense in $\ell_2(\kappa) \downarrow \alpha$ for all $\alpha \in C$.

Since X is pathological,

(8.6) $\quad E = \{\alpha \in C : X \downarrow \alpha$ is not an orthogonal direct summand of $X\}$

is stationary. By Lemma 3.5, there is $\mathbf{a}_\alpha \in X$ such that $\mathbf{a}_\alpha \downarrow \alpha \notin X \downarrow \alpha$ for all $\alpha \in E$. Let $A_\alpha = \mathrm{supp}(\mathbf{a}_\alpha)$ for $\alpha \in E$. By Lemma 8.2, we may assume without loss of generality that $E \subseteq C \cap E^\omega_\kappa$.

By FRP, there is $\alpha^* \in E^{\omega_1}_\kappa$ such that (8.3) holds for $g : E \to [\kappa]^{\aleph_0}; \alpha \mapsto A_\alpha$.

Now since $E \cap \alpha^*$ is unbounded in α^*, we have $\alpha^* \in C$. Thus $X \downarrow \alpha^*$ is non-pathological. Hence by Theorem 3.3 there are club many $I \in [\alpha^*]^{\aleph_1}$ such that $X \downarrow I$ is non-pathological and $X \downarrow I$ is dense in $\ell_2(\kappa) \downarrow I$. It follows that there is $I^* \in [\alpha^*]^{\aleph_1}$ such that

(8.7) $\quad I^*$ is closed with respect to g and closed in α^* with respect to the order topology;

(8.8) $\quad X \downarrow I^*$ is non-pathological;

(8.9) $\quad X \downarrow I^*$ is dense in $\ell_2(\kappa) \downarrow I^*$ and

(8.10) $\quad \sup(I^*) = \alpha^*$.

Let $\langle I_\alpha : \alpha < \omega_1 \rangle$ be a filtration of I^* such that $X \downarrow I_\alpha$ is dense in $\ell_2(\kappa) \downarrow I_\alpha$.

By (8.3),

(8.11) $\quad E_0 = \{\alpha \in \omega_1 : \sup(I_\alpha) \in E,\ A_{\sup(I_\alpha)} \cap \sup(I_\alpha) \subseteq I_\alpha\}$

is stationary. By Lemma 3.4 and Lemma 3.5 this is a contradiction to (8.8). This proves that FRP implies (8.1).

Since FRP is equivalent to the global negation of $\mathsf{ADS}^-(\kappa)$. Theorem 6.1 implies the converse.

For the equivalence of FRP and (8.2), it is enough by virtue of the second part of Theorem 6.1 to show that (8.1) implies (8.2).

Assume that (8.1) holds. We prove that (8.2) holds for all uncountable κ by induction on κ: if κ is $\aleph_1$ there is nothing to prove.

Suppose that $\kappa > \aleph_1$ and (8.2) has been established for all infinite cardinals $< \kappa$.

If κ is a regular cardinal then (8.2) for κ follows from (8.1), the induction hypothesis and Lemma 8.3. If κ is a singular cardinal then (8.2) for κ follows from Theorem 7.1, the induction hypothesis and Lemma 8.3. ❑ (Theorem 8.1)

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