Openly generated Boolean algebras and the Fodor-type Reflection Principle

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dedicated to Professor Dr. Sabine Koppelberg
on the occasion of her retirement

Abstract

We prove that the Fodor-type Reflection Principle (FRP) is equivalent to the assertion that any Boolean algebra is openly generated if and only if it is $\aleph_2$-projective. Previously it was known that this characterization of openly generated Boolean algebras follows from Axiom R. Since FRP is preserved by c.c.c. generic extension, we conclude in particular that this characterization is consistent with any set-theoretic assertion forcable by a c.c.c. poset starting from a model of FRP.

A crucial step of the proof of the main result is to show that FRP implies Shelah's Strong Hypothesis (SSH). In particular, we obtain that FRP implies the Singular Cardinals Hypothesis (SCH). Continuing Rinot [17], we also establish some new characterizations of SSH in terms of topological reflection theorems.

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1 Introduction

In [7], [8], [9], [12], it is shown that many mathematical reflection theorems which were originally proved under Axiom R of Fleissner [4] hold already under the Fodor-type Reflection Principle (FRP) and that most of them are even equivalent to FRP over ZFC.

The present paper deals with the reflection theorem on the open generatedness of Boolean algebras proved in [5] under Axiom R and shows that this reflection theorem is also equivalent to FRP (Theorem 5.2).

Here, FRP is the following principle introduced in [9]:

FRP : For any regular cardinal $\lambda > \aleph_1$, stationary $E \subseteq E^\lambda_\omega = \{\alpha \in \lambda : \mathrm{cf}(\alpha) = \omega\}$ and mapping $g : E \to [\lambda]^{\leq\aleph_0}$, there is $I \in [\lambda]^{\aleph_1}$ such that

(1.1) $\mathrm{cf}(I) = \omega_1$;

(1.2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap E$;

(1.3) for any regressive $f : E \cap I \to \lambda$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in E \cap I$, there is $\xi^* < \lambda$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $\sup(I)$.

In Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [9], it is shown that FRP follows from RP (see below for the definition of this principle). Since RP trivially follows from Axiom R, FRP is also a consequence of Axiom R.

In contrast to RP which implies that the size of the continuum is less or equal to $\aleph_2$ (S. Todorčević, see e.g. Theorem 37.18 in [14]), FRP does not put any restriction on the size of the continuum since FRP is preserved by c.c.c. generic extension ([9]).

RP and Axiom R are the principles defined as follows:

RP : For any cardinal λ of cofinality $> \omega_1$ and stationary $S \subseteq [\lambda]^{\aleph_0}$, there is an $I \in [\lambda]^{\aleph_1}$ such that

(1.4) $\omega_1 \subseteq I$;

(1.5) $\mathrm{cf}(I) = \omega_1$;

(1.6) $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

$\mathcal{T} \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-*club* (or *tight and unbounded* in Fleissner's terminology in [4]) if

(1.7) $\mathcal{T}$ is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and

(1.8) for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{T}$ of length ω_1, we have $\bigcup_{\alpha<\omega_1} I_\alpha \in \mathcal{T}$.

Axiom R: For any uncountable cardinal λ and stationary $S \subseteq [\lambda]^{\aleph_0}$ and ω_1-club $\mathcal{T} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{T}$ such that $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

R.E. Beaudoin [1] proved that Axiom R follows from $\mathrm{MA}^+(\sigma\text{-closed})$. It is easy to see that Axiom R implies RP. It is also easy to see that FRP implies the stationary sets of ordinals version of the reflection principles which I denote ORP:

ORP: For any cardinal λ of cofinality $> \omega_1$ and stationary $S \subseteq E^\lambda_\omega$ there is a $\delta \in E^\lambda_{\omega_1} = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \omega_1\}$ such that $S \cap \delta$ is stationary in δ.

By the remarks above and by results from [9] and [12], the axioms mentioned above can be put together in the following diagram:

$$\mathrm{MM} \Rightarrow \mathrm{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{FRP} \overset{\not\Leftarrow}{\Rightarrow} \mathrm{ORP}$$

FRP is equivalent to its seeming strengthening obtained when the phrase "there is $I \in [\lambda]^{\aleph_1}$" in the definition of FRP is replaced by "there are stationarily many $I \in [\lambda]^{\aleph_1}$":

Lemma 1.1 (FRP). *Suppose that $\lambda > \aleph_1$ is a regular cardinal. Then, for any mapping $g : E \to [\lambda]^{\leq\aleph_0}$ on a stationary $E \subseteq E^\lambda_\omega$ and club $\mathcal{C} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{C}$ such that I together with E and g satisfies* (1.1), (1.2) *and* (1.3).

Proof. Suppose that λ, E, g, $\mathcal{C}$ are as above. Let $sk_\mathcal{M}$ be the canonical Skolem-hull operator on $\mathcal{M} = \langle \mathcal{H}(\theta), \in, g, \mathcal{C}, \ldots, \trianglelefteq \rangle$ for a sufficiently large regular cardinal θ and a well-ordering $\trianglelefteq$ on $\mathcal{H}(\theta)$. Let $C^* = \{\alpha < \lambda : \omega_1 < \alpha,\ sk_\mathcal{M}(\alpha) \cap \lambda = \alpha\}$ and let $h : \lambda \to \lambda$ be defined by $h(\alpha) = \min((E \cap C^*) \setminus \alpha)$ for $\alpha \in \lambda$.

Now, let $g' : E \to [\lambda]^{\leq\aleph_0}$ be defined by

(1.9) $g'(\alpha) = g(\alpha) \cup \{h(\alpha)\}$

for $\alpha \in S$. By FRP, there is $I_0 \in [\lambda]^{\aleph_1}$ such that $\langle I_0, E, g' \rangle \models$ (1.1), (1.2), (1.3). Then, by (1.9), and since C^* is closed, we have $\sup(I_0) \in C^*$. Let $I \in [\lambda]^{\aleph_1}$ be such that $I_0 \cup \omega_1 \subseteq I \subseteq \sup(I_0)$ and $sk_\mathcal{M}(I) \cap \lambda = I$. $I \in \mathcal{C}$ since

$I = \bigcup(\mathcal{C} \cap sk_{\mathcal{M}}(I))$ by $\omega_1 \subseteq sk_{\mathcal{M}}(I)$ and elementarity. Clearly I together with E and g satisfies (1.1), (1.2) and (1.3). ❑ (Lemma 1.1)

For an uncountable set X, a *filtration* of X is a continuously $\subseteq$-increasing sequence $\langle X_\alpha : \alpha < \delta \rangle$ of subsets of X such that $\delta = \mathrm{cf}(|X|)$, $|X_\alpha| < |X|$ for all $\alpha < \delta$ and $\bigcup_{\alpha<\delta} X_\alpha = X$. If X is an algebraic structure we also assume that all X_α, $\alpha < \delta$ are subalgebras of X.

The following fact was proved in Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [9].

Lemma 1.2 (see the proof of Lemma 2.4 in [9]). *Suppose that λ, E, g are as in the definition of* FRP *and $I \in [\lambda]^{\aleph_1}$, together with these E and g, satisfies* (1.1), (1.2) *and* (1.3). *Then, for any filtration $\langle I_\alpha : \alpha < \omega_1 \rangle$ of I, the set*

$$\{\alpha < \omega_1 : \sup(I_\alpha) \in I \text{ and } g(\sup(I_\alpha)) \cap \sup(I_\alpha) \subseteq I_\alpha\}$$

is stationary in ω_1. ❑

Our notation and conventions on Boolean algebras are quite standard and follow closely S. Koppelberg [15]. In particular, a Boolean algebra B is thought to be an algebraic structure $B = \langle B, +, -, \cdot, 0, 1 \rangle$ satisfying the usual axiom of Boolean algebras with the partial ordering $\leq_B$ (or simply $\leq$ if it is clear which B is meant) defined by $a \leq_B b \Leftrightarrow a \cdot b = a$ ($\Leftrightarrow a + b = b$).

For a Boolean algebra B, let $B^+ = B \setminus \{0\}$. $X \subseteq B^+$ is pairwise disjoint if $x \cdot y = 0$ for every distinct x, $y \in B$. Recall that a Boolean algebra B is said to satisfy the c.c.c. (countable chain condition) if every pairwise disjoint $X \subseteq B^+$ is countable.

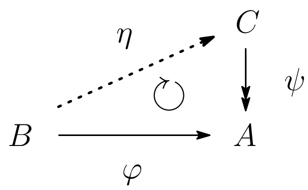
For Boolean algebras A and B, we write $A \leq B$ to denote that A is a subalgebra of B. A is a *relatively complete subalgebra* of B (notation: $A \leq_{\mathrm{rc}} B$) if $A \leq B$ and, for any $b \in B$, $p^A(b) = \sum^A A \restriction b$ exists where $A \restriction b$ denotes the ideal $\{a \in A : a \leq b\}$ on A. $p^A(b)$ is called the *lower projection* of b on A. If $A \leq_{\mathrm{rc}} B$, the *upper projection* $q^A(b)$ *of* $b \in B$ on A defined by $q^A(b) = -p^A(-b)$ is the smallest element of the filter $A \uparrow b = \{a \in A : b \leq a\}$ on A. To prove that $A \leq_{\mathrm{rc}} B$ holds, it is clearly enough to show that all $b \in B$ have their upper projection $q^A(b)$.

If $A \leq B$ but A is not a relatively complete subalgebra of B, we denote this by $A \leq_{\neg\mathrm{rc}} B$. If $A \leq B$, we have $A \leq_{\neg\mathrm{rc}} B$ if and only if there is a $b \in B$ which has no lower projection on A (i.e. the ideal $A \restriction b$ is not generated by a single element).

$A \leq B$ is a *σ-subalgebra* of B (notation: $A \leq_\sigma B$) if, for all $b \in B$, the ideal $A \restriction b$ on A is generated by countably many elements of $A \restriction b$.

For Boolean algebras A and B, $A \oplus B$ denotes the free product of A and B. Note that A and B are identified canonically with subalgebras of $A \oplus B$ and we have $A \leq_{\rm rc} A \oplus B$, $B \leq_{\rm rc} A \oplus B$ with respect to this identification.

A Boolean algebra B is *projective* if, for any Boolean algebras A and C, Boolean homomorphism $\varphi : B \to A$ and surjective Boolean homomorphism $\psi : C \to A$, there is a unique Boolean homomorphism $\eta : B \to C$ such that $\varphi = \psi \circ \eta$.



Theorem 1.3 (R. Haydon, S. Koppelberg, see [16])**.** *For a Boolean algebra B, the following are equivalent:*

(a) *B is projective.*

(b) *$B \oplus \mathrm{Fr}(\kappa)$ is free for some large enough κ.*

(c) *There is a continuously increasing chain $\langle B_\alpha : \alpha < \rho \rangle$ of subalgebras of B such that $\bigcup_{\alpha<\rho} B_\alpha = B$, B_0 is countable, $B_{\alpha+1}$ is countably generated over B_α and $B_\alpha \leq_{\rm rc} B$ for all $\alpha < \rho$.* ❑

Note that it follows from (b) above that every projective Boolean algebra satisfies the c.c.c.

A Boolean algebra B is *openly generated* (or *rc-filtered* in the terminology of L. Heindorf and L.B. Shapiro [13]) if $\{A \in [B]^{\aleph_0} : A \leq_{\rm rc} B\}$ contains a club subset of $[B]^{\aleph_0}$. The notion of the open generatedness was originally studied by E.V. Ščepin in the context of topological spaces. L. Heindorf then translated this notion into the context of Boolean algebras via Stone duality. See [13] for more historical details.

The following can be obtained immediately from the definition of the open generatedness and Theorem 1.3.

Lemma 1.4. (1) *If a Boolean algebra B is openly generated and $|B| \leq \aleph_1$, then B is projective.*

(2) *A Boolean algebra B is openly generated if and only if, for any σ-closed p.o. $\mathbb{P}$ forcing $|B|$ to be less or equal to $\aleph_1$, we have $\Vdash_{\mathbb{P}}$ "B is projective".*

(3) *If a Boolean algebra B is projective then B is openly generated.*

(4) *An openly generated Boolean algebra B satisfies the c.c.c.* ❑

Let κ be a cardinal and P a property of a Boolean algebra. A Boolean algebra B is said to be κ-P if the set

$$\{C \in [B]^{<\kappa} : C \leq B \text{ and } C \text{ satisfies } P\}$$

contains a club subset of $[B]^{<\kappa}$.

By Lemma 1.4, (1) and (3), we obtain:

Lemma 1.5. *A Boolean algebra B is $\aleph_2$-projective if and only if B is $\aleph_2$-openly generated.* ❑

Our main theorem can be now formulated as follows:

Theorem 1.6. *Assume* FRP*. Then, for any Boolean algebra B, the following are equivalent:*

(a) *B is openly generated.*

(b) *B is $\aleph_2$-projective (i.e. $\aleph_2$-openly generated).*

We shall prove this theorem in Section 4. The proof of Theorem 1.6 uses the fact that FRP implies Shelah's Strong Hypothesis (SSH). This fact is established in the following Section 2. In continuation of Rinot [17], we also give some new characterizations of SSH in terms of topological reflection theorems.

In Section 3, we give a fairly self-contained exposition of (mostly already known) results on openly generated and $\aleph_2$-openly generated Boolean algebras which we need in the proof of Theorem 1.6.

The assertion of Theorem 1.6 was proved first under $\mathrm{MA}^+(\sigma\text{-closed})$ by the first author in [5]. Qi Feng then pointed out that almost the same proof can be applied to prove the assertion under Axiom R (see also Fuchino [6]).

On the other hand, the proof of Theorem 1.6 is not a straightforward modification of the proof under Axiom R in [5]. This is partially due to the fact that FRP cannot handle with the reflection on any singular cardinal λ (see Lemma 2.2 in [9]). But, even for regular $\lambda > \aleph_1$, it appears that we need some more algebraic tools (some being proved under SSH) from Section 3.

In [5], a counterexample to the assertion of Theorem 1.6 was constructed under the existence of a non-reflecting stationary set $S \subseteq E^{\kappa}_{\omega}$ for some regular cardinal κ. This means that the assertion of Theorem 1.6 implies

ORP. In Section 5, we construct a counterexample to the assertion of Theorem 1.6 under the existence of an almost essentially disjoint ladder system $g : S \to [\kappa]^{\aleph_0}$ for a regular cardinal κ and a stationary $S \subseteq E^{\kappa}_{\omega}$. In Fuchino, Sakai, Soukup and Usuba [12], it is proved that the existence of such a ladder system is equivalent to the negation of FRP. Thus, this construction shows that the assertion of Theorem 1.6 implies FRP and hence it is equivalent to FRP over ZFC (Theorem 5.2).

2 FRP implies Shelah's Strong Hypothesis

In light of previous works showing that the Singular Cardinal Hypothesis (SCH) follows from diverse reflection principles (see e.g. [20], [21], [22], [23]), it seems natural to ask if FRP also implies SCH.

However, in contrast to the principles considered in the papers cited above, FRP does not imply that the continuum is very small. Hence the right question to be asked here seems to be rather if FRP implies Shelah's Strong Hypothesis (SSH). Note that SCH and SSH become equivalent under $2^{\aleph_0} < \aleph_\omega$ (see Theorem 2.1 and (2.2) below).

In this section, we shall give the positive answer to the latter question. Of course the positive answer to the former question follows from this.

Let us begin with reviewing the definition and some basic facts about SSH. Shelah's Strong Hypothesis (SSH) states that the pseudopowers of singular cardinals take their minimal possible values, i.e. $\mathrm{pp}(\lambda) = \lambda^+$ holds for all singular λ. Here, the pseudopower $\mathrm{pp}(\lambda)$ is defined as the supremum of

$$\begin{aligned}\mathrm{PP}(\lambda) = \{\mathrm{cf}(\textstyle\prod \mathbf{a}/D) \,:\, & \mathbf{a} \subseteq \lambda \cap \mathit{Reg}, \sup \mathbf{a} = \lambda, \mathrm{otp}(\mathbf{a}) = \mathrm{cf}(\lambda),\\ & D \text{ is an ultrafilter over } \mathbf{a} \text{ disjoint from the}\\ & \text{ideal } I_b(\mathbf{a}) \text{ of the bounded subsets of } \mathbf{a}\,\}.\end{aligned}$$

Claim 2.4 in Section II of [19] shows that a result similar to Silver's theorem holds for $\mathrm{pp}(\cdot)$. From this it follows that

(2.1) if $\mathrm{pp}(\lambda) > \lambda^+$ for some singular cardinal λ then there is a singular cardinal $\lambda' \leq \lambda$ of countable cofinality such that $\mathrm{pp}(\lambda') > (\lambda')^+$.

SSH is actually a cardinal arithmetical statement.

Theorem 2.1 (S. Shelah, see [17]). SSH *is equivalent to each of the following assertions:*

(a) $\mathrm{cf}([\kappa]^\theta, \subseteq) = \kappa$ *holds for all cardinals* κ, θ *with* $\theta < \mathrm{cf}(\kappa)$.

(b) $\mathrm{cf}([\kappa]^{\aleph_0}, \subseteq) = \kappa^+$ *for all singular cardinals* κ, *with* $\mathrm{cf}(\kappa) = \omega$. ❑

Note that the implication (b) ⇒ SSH in Theorem 2.1 follows from (2.1).

By Silver's theorem, it is easy to see that the Singular Cardinal Hypothesis (SCH) is equivalent to the assertion:

(2.2) $\mathrm{cf}([\kappa]^{\aleph_0}, \subseteq) = \kappa^+$ for every singular cardinal $\kappa > 2^{\aleph_0}$ of cofinality ω.

From this and the characterization of SSH above, it is clear that SSH implies SCH.

S. Shelah (Claim 1.3 in Chapter II of [19]) proved that

(2.3) if $\mathrm{pp}(\lambda) > \lambda^+$ for a singular cardinal λ, then there is a better scale $\langle \vec{\lambda}, \vec{f} \rangle$ for λ.

Here, a pair $\langle \vec{\lambda}, \vec{f} \rangle$ is said to be a *better scale* for a singular cardinal λ if

(2.4) $\vec{\lambda} = \langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ is a strictly increasing sequence of regular cardinals cofinal in λ;

(2.5) $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$, $f_\alpha \in \prod_{i<\mathrm{cf}(\lambda)} \lambda_i$ for all $\alpha < \lambda^+$ and $\langle f_\alpha : \alpha < \lambda^+ \rangle$ is a scale in $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$ with respect to the ideal $I_b(\mathrm{cf}(\lambda))$ of the bounded subsets of $\mathrm{cf}(\lambda)$; and

(2.6) For every $\delta < \lambda^+$ with $\mathrm{cf}(\delta) > \mathrm{cf}(\lambda)$, there is a closed unbounded $C \subseteq \delta$ such that

(2.6a) $\mathrm{otp}(C) = \mathrm{cf}(\delta)$ and

(2.6b) for all $\beta \in C$ there is $i < \mathrm{cf}(\lambda)$ such that $f_\gamma(j) < f_\beta(j)$ for all $j \geq i$ and $\gamma \in C \cap \beta$.

It is proved in Cummings, Foreman and Magidor [2, Theorem 4.1] that the existence of a better scale for singular λ implies the combinatorial principle ADS_λ. Here,

ADS_λ: There exists a sequence $\langle a_\alpha : \alpha < \lambda^+ \rangle$ of unbounded subsets of λ such that

(2.7) $\mathrm{otp}(a_\alpha) = \mathrm{cf}(\lambda)$; and

(2.8) for all $\beta < \lambda^+$ there exists $g : \beta \to \kappa$ such that the sequence $\langle a_\alpha \setminus g(\alpha) : \alpha < \beta \rangle$ consists of pairwise disjoint sets.

In [9] it is shown that ADS_λ for a singular cardinal λ of cofinality ω implies $\mathrm{ADS}^-(\lambda^+)$ where, for a regular cardinal κ,

$\mathrm{ADS}^-(\kappa)$: There are a stationary set $S \subseteq \kappa$ and $g : S \to [\kappa]^{\aleph_0}$ such that

(2.9) $g(\alpha) \subseteq \alpha$ and $\mathrm{otp}(g(\alpha)) = \omega$ for all $\alpha \in S$;

(2.10) g is almost essentially disjoint. That is, for all $\beta < \kappa$, there is a function $f : S \cap \beta \to [\kappa]^{<\aleph_0}$ such that $f(\alpha) < \sup(g(\alpha))$ for all $\alpha \in S \cap \beta$ and $g(\alpha) \setminus f(\alpha)$, $\alpha \in S \cap \beta$ are pairwise disjoint.

It is also shown in [9] that $\mathrm{ADS}^-(\kappa)$ for a regular uncountable κ implies $\neg$FRP — actually, we can further show that FRP is equivalent to the statement that $\mathrm{ADS}^-(\kappa)$ does not hold for all regular uncountable κ (see [12, Theorem 2.5]).

By putting together the facts mentioned above we obtain a proof of the following Theorem 2.2[1)]. We give here a slightly more direct proof of this theorem.

Theorem 2.2. FRP *implies* SSH.

Proof. Suppose $\neg$SSH. Then, by (2.1), there is a singular cardinal λ such that $\mathrm{cf}(\lambda) = \omega$ and $\mathrm{pp}(\lambda) > \lambda^+$. By (2.3), there is a better scale $\langle \vec{\lambda}, \vec{f} \rangle$ for λ, say, with $\vec{\lambda} = \langle \lambda_i : i < \omega \rangle$ and $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$. Fix a one-to-one mapping $\varphi : {}^{\omega>}\lambda \to \lambda$ and let $E = E^{\lambda^+}_\omega \setminus \lambda$. Let $g : E \to [\lambda^+]^{\aleph_0}$ be defined by

$$(2.11) \quad g(\alpha) = \{\varphi(f_\alpha \restriction n) : n \in \omega\} \text{ for } \alpha \in E.$$

Note that we actually have $g : E \to [\lambda]^{\aleph_0}$ and hence $g(\alpha) \subseteq \alpha$ for all $\alpha \in E$.

We show that this g together with the E as above is a counterexample to FRP.

Suppose that $I \in [\lambda]^{\aleph_1}$ satisfies (1.1) and (1.2). We have to show that I does not satisfy (1.3).

Let $\delta = \sup(I)$. Then there is a club $C \subseteq \delta$ satisfying (2.6a) and (2.6b). For $n \in \omega$, let

$$(2.12) \quad E_n = \{\beta \in C : f_\gamma(j) < f_\beta(j) \text{ holds for all } \gamma \in C \cap \beta \text{ and } j \geq n, \\ n \text{ is the minimal number with this property for } \beta\}.$$

By (2.6b), we have

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1) After the results in this section has been obtained, the authors learned that Toshimichi Usuba was also aware of Theorem 2.2 by the same combination of the known results.

(2.13) $C = \dot{\bigcup}_{n\in\omega} E_n$.

Let $f : E \cap I \to \lambda$ be defined by

$$(2.14)\quad f(\alpha) = \begin{cases} \varphi(f_\alpha \restriction (n+1)), & \text{if } \alpha \in E_n \\ \varphi(\emptyset), & \text{if there is no } n \text{ as above.} \end{cases}$$

Then, we have $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in E \cap I$. For any stationary $S \subseteq E \cap \delta$, there is $n^* \in \omega$ such that $S \cap E_{n^*}$ is stationary by (2.13). For α, $\alpha' \in S \cap E_{n^*}$ with $\alpha < \alpha'$ we have $f_\alpha(n^*) < f_{\alpha'}(n^*)$ by (2.12). It follows that $f(\alpha) \neq f(\alpha')$ for all α, $\alpha' \in S \cap E_{n^*}$ by the definition (2.14) of f.

This shows that the mapping f as above exemplifies the failure of (1.3) for this I. ❑ (Theorem 2.2)

As we already mentioned in the introduction, FRP is known to be equivalent to many mathematical reflection theorems. Hence the implication of SSH from FRP suggests that SSH can be also regarded as a reflection theorem. In fact, Rinot [17] as well as the next Theorems 2.3 and 2.4 show that SSH can be characterized in terms of topological reflection theorems.

The proof of the following theorems is a further development of the idea of proof of the Theorem in [17] and is also similar but much more involved than the one for the proof of the implication of SSH from FRP given above.

Let us begin with the definition of the topological notions used in the next theorem. A topological space X is said to be *thin* if, for any $D \subseteq X$, we have $|\,\overline{D}\,| \leq |\,D\,|^+$. For a cardinal κ, X is said to be $< \kappa$*-thin* if, for any $D \in [X]^{<\kappa}$ we have $|\,\overline{D}\,| \leq |\,D\,|^+$.

Recall that a topological space X is countably tight if for any $Y \subseteq X$ and $x \in X$, we have $x \in \overline{Y}$ if and only if there is $Y' \in [Y]^{\aleph_0}$ such that $x \in \overline{Y'}$. The density $d(X)$ of a topological space X is the minimal size of $D \subseteq X$ such that $\overline{D} = X$.

Theorem 2.3. *The following are equivalent:*

(a) SSH.

(b) *For any countably tight topological space X, if X is $< \aleph_1$-thin, then X is thin.*

(c) *For any countably tight topological space X, if X is $< \kappa$-thin for $\kappa = \max\{\aleph_1, d(X)\}$, then X is thin.*

Proof. (a) $\Rightarrow$ (b): Assume SSH and suppose that X is a countably tight $< \aleph_1$-thin topological space.

Let $D \subseteq X$. By SSH, there is an $H \subseteq [D]^{\aleph_0}$ of cardinality $\leq |D|^+$ such that H is cofinal in $[D]^{\aleph_0}$ with respect to $\subseteq$. By countable tightness, we have $\overline{D} = \bigcup\{\overline{Y} : Y \in H\}$. Since X is $< \aleph_1$-thin, we have $|\overline{Y}| \leq \aleph_1$ for all $Y \in H$. Thus $|\overline{D}| = |\bigcup\{\overline{Y} : Y \in H\}| \leq |H| \cdot \aleph_1 \leq |D|^+$.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a): Assume $\neg$SSH. Then by (2.1) there is a singular cardinal λ of countable cofinality such that $\mathrm{pp}(\lambda) > \lambda^+$. Let $\langle \lambda_n : n < \omega \rangle$ be an increasing sequence of regular cardinals cofinal in λ and $\mathcal{D}$ an ultrafilter over ω such that $\mathcal{D}$ is disjoint from the ideal $I_b(\omega)$ of the bounded subsets of ω (i.e. $\mathcal{D}$ is non-principal) and $\mathrm{cf}(\prod_{i<\omega} \lambda_i, <_{\mathcal{D}}) \geq \lambda^{++}$.

Let $\kappa = \lambda^+$. Since κ is regular, we have $E^{\kappa^+}_{<\kappa} \in I[\kappa^+]$ by Lemma 4.4 in [18]. By Lemma 2.3 in [18], this means that there is a sequence $\vec{x} = \langle x_\alpha : \alpha < \kappa^+ \rangle$ with $x_\alpha \subseteq \alpha$ for all $\alpha < \kappa^+$ and a club $C \subseteq \kappa^+$ such that

(2.15) for every $\delta \in E^{\kappa^+}_{<\kappa} \cap C$, there is a cofinal subset a_δ of δ of order type $\mathrm{cf}(\delta)$ such that $a_\delta \cap \alpha = x_\alpha$ for all $\alpha \in a_\delta$.

We define now a $<_{\mathcal{D}}$ increasing sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ of elements of $\prod_{i<\omega} \lambda_i$ inductively as follows: Let $f_0 \in \prod_{i<\omega} \lambda_i$ be arbitrary. Suppose that $\langle f_\beta : \beta < \alpha \rangle$ has been chosen for some $\alpha < \kappa^+$. Since $\mathrm{cf}(\prod_{i<\omega} \lambda_i, <_{\mathcal{D}}) \geq \kappa^+$, there is some $g \in \prod_{i<\omega} \lambda_i$ such that $f_\beta <_{\mathcal{D}} g$ for all $\beta < \alpha$. Let $f_\alpha \in \prod_{i<\omega} \lambda_i$ be defined by

$$(2.16)\quad f_\alpha(n) = \begin{cases} \sup(\{f_\beta(n) : \beta \in x_\alpha\} \cup \{g(n)\}), & \text{if } |x_\alpha| < \lambda_n\,, \\ g(n), & \text{otherwise.} \end{cases}$$

Since $g \leq f_\alpha$ (coordinatewise), we have $f_\beta <_{\mathcal{D}} f_\alpha$ for all $\beta < \alpha$.

Fix a one-to-one mapping $\psi : {}^{\omega>}\lambda \to \lambda \setminus \omega_1$ and let $\mathcal{F} = \{y_\alpha : \alpha \in C\}$ where $y_\alpha = \{\psi(f_\alpha \restriction n) : n \in \omega\}$ for $\alpha < \kappa^+$. Note that $\mathcal{F} \subseteq [\lambda \setminus \omega_1]^{\aleph_0}$ and hence $\mathcal{F} \cap \lambda = \emptyset$.

Let X be the the Mrówka space over disjoint union of $\mathcal{F}$ and λ, that is, the space $X \cup \lambda$ with the topology τ such that each $\alpha < \lambda$ is isolated and each $y \in \mathcal{F}$ has the neighborhood base

$$\mathcal{B}_y = \{\{y\} \cup (y \setminus s) : s \in [y]^{<\aleph_0}\}.$$

$\langle X, \tau \rangle$ is then first countable and hence countably tight.

We show that $\langle X, \tau \rangle$ is a counterexample to (c). Note that, for any $A \subseteq \lambda$, we have

$$(2.17)\quad \overline{A} = A \cup \{y \in \mathcal{F} : |A \cap y| = \aleph_0\}$$

and, for any $B \subseteq X$, we have

(2.18) $\lambda \cap \overline{B} = \lambda \cap B$.

In particular, we have $\overline{\lambda} = X$ and $d(X) = \lambda$. Since $\lambda^+ < \kappa^+ = |\, X \,| = |\, \overline{\lambda} \,|$, X is not thin. Thus the following claim establishes that X is a counterexample to (c).

Claim 2.3.1. *For every $D \in [X]^{<\lambda}$ we have $|\, \overline{D} \,| = |\, D \,|$.*

$\vdash$ Suppose not and let $D \in [X]^{<\lambda}$ be such that $|\, \overline{D} \,| > |\, D \,|$. Since $\lambda \cap \overline{D} = \lambda \cap D$ by (2.18), we have

(2.19) $|\, \mathcal{F} \cap \overline{D} \,| > |\, D \,|$.

By (2.17), it follows that $|\, \lambda \cap D \,| \geq \aleph_0$. Let $\theta = |\, D \,|^+$. By (2.19), there is $I \in [C]^\theta$ such that $\mathrm{otp}(I) = \theta$ and $y_\alpha \in \overline{D}$ for all $\alpha \in I$. Let $\delta = \sup(I)$. Then $\mathrm{cf}\,\delta = \theta < \lambda < \kappa$. Since C is closed, we have $\delta \in C$. Thus $\delta \in E^{\kappa^+}_{<\kappa} \cap C$ and there is an a_δ as in (2.15) for this δ.

Let $j_\gamma \in a_\delta$ and $i_\gamma \in I$ for $\gamma < \theta$ be defined inductively such that

(2.20) $j_{\gamma_0} < i_{\gamma_0} < j_{\gamma_1} < i_{\gamma_1}$ for all $\gamma_0 < \gamma_1 < \theta$.

In particular we have $j_\gamma < i_\gamma < j_{\gamma+1}$ for all $\gamma < \theta$ and hence $f_{j_\gamma} <_{\mathcal{D}} f_{i_\gamma} <_{\mathcal{D}} f_{j_{\gamma+1}}$.

Let $n^* \in \omega$ be such that $\lambda_{n^*} > \theta$. Since $\mathcal{D}$ is a filter and disjoint from $I_b(\omega)$, for each $\gamma < \theta$, there is an $n_\gamma \in \omega \setminus n^*$ such that

(2.21) $f_{j_\gamma}(n_\gamma) < f_{i_\gamma}(n_\gamma) < f_{j_{\gamma+1}}(n_\gamma)$.

For $n < \omega$, let $I(n) = \{\gamma < \theta \,:\, n_\gamma = n\}$. Then we have $\theta = \bigcup_{n<\omega} I(n)$. Since $\mathrm{cf}(\theta) = \theta > \omega$, there is $n^\dagger < \omega$ such that $I(n^\dagger)$ is cofinal in θ.

Subclaim 2.3.1.1. *$\langle f_{i_\gamma}(n^\dagger) \,:\, \gamma \in I(n^\dagger)\rangle$ is strictly increasing.*

$\vdash$ For $\gamma_0, \gamma_1 \in I(n^\dagger)$ with $\gamma_0 < \gamma_1$, we have $i_{\gamma_0} < j_{\gamma_0+1} \leq j_{\gamma_1} < i_{\gamma_1}$. Hence, we have $f_{i_{\gamma_0}}(n^\dagger) < f_{j_{\gamma_0+1}}(n^\dagger)$ and $f_{j_{\gamma_1}}(n^\dagger) < f_{i_{\gamma_1}}(n^\dagger)$.

Thus it is enough to show

(2.22) $f_{j_{\gamma_0+1}}(n^\dagger) \leq f_{j_{\gamma_1}}(n^\dagger)$.

If $\gamma_0 + 1 = \gamma_1$, this is trivial. If $j_{\gamma_0+1} < j_{\gamma_1}$, then we have $j_{\gamma_0+1} \in a_\delta \cap j_{\gamma_1}$ by the choice of j_γ's. Since $x_{j_{\gamma_1}} = a_\delta \cap j_{\gamma_1}$ by (2.15), we have $|\, s_{j_{\gamma_1}} \,| \leq |\, a_\delta \,| = \theta < \lambda_{n^*} \leq \lambda_{n^\dagger}$. Thus the first case in (2.16) has been applied when

$f_{j_{\gamma_1}}(n^\dagger)$ was defined. In particular, since $j_{\gamma_0} \in a_\delta \cap j_{\gamma_1} = x_{j_{\gamma_1}}$, (2.22) holds.
$\dashv$ (Subclaim 2.3.1.1)

Let $I' = \{i_\gamma : \gamma \in I(n^\dagger)\}$. Since $I' \subseteq I$ we have $\{y_\alpha : \alpha \in I'\} \subseteq \overline{D}$. It follows by (2.17) that $y_\alpha \cap D$ is infinite for all $\alpha \in I'$. Since $\langle f_\alpha(n^\dagger) : \alpha \in I' \rangle$ is strictly increasing sequence of length θ,

$$\{(D \cap y_\alpha) \setminus \{\psi(f_\alpha \restriction m) : m < n^\dagger\} : \alpha \in I'\}$$

is a family of θ many pairwise disjoint infinite subsets of D. This is a contradiction to $|D| < \theta$. $\dashv$ (Claim 2.3.1)
❑ (Theorem 2.3)

We also obtain the following Theorem 2.4 by almost the same proof as above.

Let us call a topological space X *very thin* if, for every $D \subseteq X$ of regular cardinality $|\overline{D}| = |D|$ holds. Let us also say that X is *$< \kappa$-very thin* if, for every $D \subseteq X$ of regular cardinality $< \kappa$, $|\overline{D}| = |D|$ holds.

Note that if a very thin space X is countably tight then we also have $|\overline{D}| = |D|$ for all $D \subseteq X$ of cardinality with uncountable cofinality.

Theorem 2.4. *The following are equivalent:*

(a) SSH.

(b') *For any countably tight topological space X, if X is $< \aleph_1$-very thin, then X is very thin.*

(c') *For any countably tight topological space X, if X is $< \kappa$-very thin for $\kappa = \max\{\aleph_1, d(X)\}$, then X is very thin.* ❑

Theorem in [17] is now almost included in Theorem 2.4.

Corollary 2.5 (A version of (1) $\Leftrightarrow$ (4) of Theorem in Rinot [17])**.** *The following are equivalent:*

(a) SSH.

(d) *For a countably tight topological space X, if*

$$(\mathrm{cf}(d(X)) > \omega \text{ and } |X| > d(X)) \quad \text{or} \quad |X| > d(X)^+,$$

then there is a countable subset D of X such that $|\overline{D}| = |X|$.

Proof. "(a) $\Rightarrow$ (d)" can be shown by the proof of "(a) $\Rightarrow$ (b)" of Theorem 2.3 with the infinite version of pigeonhole principle.

"(d) $\Rightarrow$ (a)" follows from "(b') $\Rightarrow$ (a)" of Theorem 2.4. ❑ (Corollary 2.5)

3 Properties of openly generated and $\aleph_2$-openly generated Boolean algebras

In this section, we consider some results on openly generated and $\aleph_2$-openly generated Boolean algebras needed in the next section.

Lemma 3.10 and Theorem 3.12 below appeared already in [6]. Under Axiom R, Theorem 1.6 would immediately follow from these results. However, for our proof of Theorem 1.6 under FRP, we need apparently some more structure theory on $\aleph_2$-openly generated Boolean algebras.

[6] provides assertions (Corollary A.4.6 and Theorem A.4.7 in [6]) on $\aleph_2$-openly generated Boolean algebras beyond our Lemma 3.10 and Theorem 3.12 which could be used for our purpose. Unfortunately, for the proof of these assertions in [6], a lemma is used which was later proved to be independent from ZFC (under some large cardinal) and whose proof (see Theorem 10 in [11]) requires a very weak form of square principle and the status of this square principle under Axiom R or even FRP is still open (see also "added in proof" in [10]).

To avoid this problem, we make use of Theorem 3.13 below in the proof of Theorem 1.6 in Section 4.

For a Boolean algebra B, a mapping $f : B \to [B]^{<\aleph_0}$ is called a *Freese-Nation mapping* (*FN-mapping*, for sort) on B if, for any $a, b \in B$ with $a \leq b$, there is $c \in f(a) \cap f(b)$ such that $a \leq c \leq b$. The following is the essence of Theorem 3.2, 3.2. We sketch the proof for the convenience of the reader.

Lemma 3.1 (L. Heindorf and L.B. Shapiro [13]). (1) *Suppose that B is an openly generated Boolean algebra with a closed unbounded $\mathcal{C} \subseteq \{C \in [B]^{\aleph_0} : C \leq_{\rm rc} B\}$. If $D \leq B$ is such that $\mathcal{C} \cap [D]^{\aleph_0}$ is closed unbounded in $[D]^{\aleph_0}$, then $D \leq_{\rm rc} B$.*

(2) *Suppose that B is a Boolean algebra and $f : B \to [B]^{<\aleph_0}$ is an FN-mapping. If $A \leq B$ is closed with respect to f then we have $A \leq_{\rm rc} B$.*

(3) *Suppose that $A \leq_{\rm rc} B$ and both of A and B have FN-mapping. Then for any FN-mapping f on A there is an FN-mapping $\tilde{f}$ on B extending f.*

Proof. (1): Suppose that $D \leq B$ and $\mathcal{C} \cap [D]^{\aleph_0}$ is closed unbounded in $[D]^{\aleph_0}$. If $D \leq_{\neg{\rm rc}} B$ then there would be a $b \in B$ without its lower projection into D. Then we can construct a continuously increasing sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{C} \cap [D]^{\aleph_0}$ such that $\langle p^{C_\alpha}(b) : \alpha < \omega_1 \rangle$ is strictly increasing. But this is a contradiction to Lemma 1.4, (4).

(2): Suppose that $A \leq B$ is closed with respect to an FN-mapping $f : B \to [B]^{<\aleph_0}$. Then, for arbitrary $b \in B$, we have $p^A(b) = \sum^A f(b) \cap A$.

(3): Let $g : B \to [B]^{<\aleph_0}$ be an FN-mapping. Then the mapping $\tilde{f}$ on B defined by

$$\tilde{f}(b) = \begin{cases} f(b), & \text{if } b \in A; \\ g(b) \cup f(p^A(b)) \cup f(q^A(b)), & \text{otherwise} \end{cases}$$

for $b \in B$ is as desired. ❑ (Lemma 3.1)

The equivalence of (a), (b), (c) of the following theorem can be proved using Lemma 3.1, (1), (3) while the implication from (c) to (d) follows from Lemma 3.1, (2) and the implication from (e) to (a) follows immediately from the definition of open generatedness.

Theorem 3.2. *For a Boolean algebra B, the following are equivalent.*

(a) *B is openly generated.*

(b) (Heindorf and Shapiro [13]) *There is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ of B with $\lambda = |\, B \,|$ such that, for every $\alpha < \lambda$, $B_\alpha \leq_{\rm rc} B$, $|\, B_\alpha \,| = |\, \alpha + \omega \,|$ and B_α is openly generated.*

(c) (Heindorf and Shapiro [13]) *There is a FN-mapping on B.*

(d) (Fuchino [6]) *For any sufficiently large regular θ and $M \prec \mathcal{H}(\theta)$ with $B \in M$, we have $B \cap M \leq_{\rm rc} B$.*

(e) (Fuchino [6]) *For any sufficiently large regular θ and countable $M \prec \mathcal{H}(\theta)$ with $B \in M$, we have $B \cap M \leq_{\rm rc} B$.* ❑

Theorem 3.2, (a)⇔(c) and Lemma 3.1, (3) implies the following:

Theorem 3.3 (Heindorf and Shapiro [13])**.** *Suppose that $\langle B_\alpha : \alpha < \delta \rangle$ is a continuously increasing sequence of openly generated Boolean algebras for some limit ordinal δ such that $B_\alpha \leq_{\rm rc} B_{\alpha+1}$ for every $\alpha < \delta$. Then $B = \bigcup_{\alpha<\delta} B_\alpha$ is also openly generated.* ❑

Theorem 3.4 (I. Bandlow, unpublished, see Theorem 2.2.11 in [13])**.** *If B_n, $n \in \omega$ are openly generated Boolean algebras such that $B_n \leq_\sigma B_{n+1}$ for all $n \in \omega$, then $B = \bigcup_{n\in\omega} B_n$ is also openly generated.*

Proof. We give here a proof using the characterization Theorem 3.2, (a)⇔(e) of openly generated Boolean algebras.

Let θ be a sufficiently large regular cardinal. It is enough to show that, for any countable $M \prec \mathcal{H}(\theta)$ with $B \in M$, we have $B \cap M \leq_{\rm rc} B$.

Without loss of generality, we may assume that $\langle B_n : n \in \omega \rangle \in M$. Suppose that $b \in B$. Then $b \in B_{n^*}$ for some $n^* \in \omega$. Since B_{n^*} is openly generated and $B_{n^*} \in M$, $b^* = p^{B_{n^*} \cap M}(b)$ exists by Theorem 3.2, (a)⇔(e).

We show that $b^* = p^{B \cap M}(b)$. Suppose $b' \in (B \cap M) \restriction b$. Since $B_{n^*} \leq_\sigma B$, there is a countable $X \subseteq B_{n^*}$ coinitial in $B_{n^*} \uparrow b' = \{c \in B_{n^*} : b' \leq c\}$. We may assume that $X \in M$ and hence $X \subseteq M$ by the countability of X. Since $b \in B_{n^*} \uparrow b'$, there is $b'' \in X \subseteq B_{n^*} \cap M$ such that $b'' \leq b$. Since $b'' \in B \cap M \restriction b$, we have $b' \leq b'' \leq b^*$. ❑ (Lemma 3.4)

The following Lemma should be a folklore:

Lemma 3.5. *For a regular cardinal κ and a club $C \subseteq [X]^\kappa$ for a some set X with $\kappa \subseteq X$, there is a mapping $f : X^{<\omega} \to X$ such that*

$$C(f) = \{a \in [X]^\kappa : \kappa \subseteq a \text{ and } a \text{ is closed with respect to } f\} \subseteq C.$$

Proof. Let θ be sufficiently large. We may assume that $\kappa < |X|$. Let $\mathcal{M} = \langle \mathcal{H}(\theta), \in, X, C, \trianglelefteq \rangle$ where $\trianglelefteq$ is a well-ordering on $\mathcal{H}(\theta)$. Let $N = sk_{\mathcal{M}}(|X|)$ where $sk_{\mathcal{M}}(\cdot)$ denotes the Skolem-hull operator corresponding to the built-in Skolem functions of $\mathcal{M}$. Let $\varphi : X \to N$ be a bijection and let $f : N^{<\omega} \to N$ code the built-in Skolem functions of $\mathcal{M} \restriction N$ and φ^{-1}.

Now, identifying $\langle N, \{\varphi''c : c \in C\} \rangle$ with $\langle X, C \rangle$, we can show that this f is as desired:

Claim 3.5.1. *If $a \subseteq N$ is closed with respect to f and $\kappa \subseteq a$ then $a \cap X \in C$ and $a = \varphi''a \cap X$.*

⊢ Since a is closed with respect to the Skolem functions, we have $a \prec N$ (i.e. $\mathcal{M} \restriction a \prec \mathcal{M} \restriction N$). Since $\kappa \subseteq a$, it follows that $a \cap X = \bigcup(a \cap C)$ and $a \cap C$ is upward directed by the elementarity. Thus $a \cap X \in C$ by closedness of C. Since a is also closed with respect to φ^{-1}, we have $a = \varphi''a \cap X$.

⊣ (Claim 3.5.1)

❑ (Lemma 3.5)

Note that we cannot drop the condition "$\kappa \subseteq a$" for an uncountable κ in general (see Feng [3]).

Lemma 3.6. *Suppose that X is a set and κ is a regular cardinal $< |X|$ and $C \subseteq [X]^\kappa$ is club in $[X]^\kappa$. Then, for $\lambda = |X|$,*

$$\mathcal{C} = \{Y \in [X]^{<\lambda} : C \cap [Y]^\kappa \text{ is club in } [Y]^\kappa\}$$

contains a set $\mathcal{C}'$ such that, for all regular λ' with $\kappa < \lambda' \leq \lambda$, $\mathcal{C}' \cap [X]^{<\lambda'}$ is club in $[X]^{<\lambda'}$.

Proof. We may assume that $\kappa \subseteq X$. By Lemma 3.5, there is a mapping $f : X^{<\omega} \to X$ such that

$$C(f) = \{a \in [X]^{\kappa} : \kappa \subseteq a \text{ and } a \text{ is closed with respect to } f\} \subseteq C.$$

Let

$$\mathcal{C}' = \{Y \in [X]^{<\lambda} : \kappa \subseteq Y \text{ and } Y \text{ is closed with respect to } f\}.$$

Then $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{C}'$ is as desired. ❑ (Lemma 3.6)

Proposition 3.7. *Suppose that B is a κ-openly generated Boolean algebra for a regular cardinal κ. If θ is a sufficiently large regular cardinal and $M \prec \mathcal{H}(\theta)$ is such that $B \in M$ and $\kappa \leq |\,M\,| \subseteq M$, then $B \cap M$ is also a κ-openly generated Boolean algebra.*

Proof. If $|\,M\,| \geq |\,B\,|$ then $B \cap M = B$ and hence the assertion trivially holds. So we may assume $|\,M\,| < |\,B\,|$. Let $C = \{A \in [B]^{<\kappa} : A \text{ is openly generated}\}$. Since C contains a club subset of $[B]^{<\kappa}$, there are $\mathcal{C}$ and $\mathcal{C}'$ as in Lemma 3.6 for this C. By the elementarity of M, we may assume that $\mathcal{C}' \in M$. Then we have $B \cap M = \bigcup \mathcal{C}' \cap M \in \mathcal{C}'$. In particular, $B \cap M$ is κ-openly generated. ❑ (Proposition 3.7)

For a Boolean algebra B, $X \subseteq B$ and $b \in B$, let $tp_X(b) = \langle X \restriction b, X \uparrow b\rangle$. Let us say that B is *ω-stable* if $|\,\{tp_X(b) : b \in B\}\,| \leq \aleph_0$ for all $X \in [B]^{\aleph_0}$. Note that ω-stability defined here only roughly corresponds to the model theoretic notion of the ω-stability of structures. Clearly B is ω-stable if $\{tp_A(b) : b \in B\}$ is countable for cofinally many countable $A \leq B$ (where "cofinally many" refers to the cofinality in $[B]^{\aleph_0}$ with respect to $\subseteq$). If $A \leq B$, A is countable and $A \leq_{\mathrm{rc}} B$ then each $tp_A(b)$ is decided by $\langle p^A(b), q^A(b)\rangle$ and hence we have $|\,\{tp_A(b) : b \in B\}\,| \leq \aleph_0$. Thus:

Lemma 3.8. *If a Boolean algebra B is projective then B is ω-stable.* ❑

For a Boolean algebra B and $X \subseteq B$, let $X^{\perp} = \{c \in B : b \cdot c = 0 \text{ for every } b \in X\}$. An ideal I on B is said to be regular if $(I^{\perp})^{\perp} = I$. Note that $(X^{\perp})^{\perp} \supseteq X$ holds for any $X \subseteq B$. In the following, we shall also simply write $I^{\perp\perp}$ in place of $(I^{\perp})^{\perp}$.

A Boolean algebra B is said to have the *Bockstein Separation Property* (*BSP* for short) if every regular ideal I of B is countably generated, i.e. if there is always a countable cofinal subset of such I.

Theorem 3.9 (Koppelberg [16])**.** *If a Boolean algebra B is projective then B has the BSP.* ❑

For a regular uncountable cardinal θ, $M \prec \mathcal{H}(\theta)$ is said to be *ω-bounding* if for every $x \in [M]^{\aleph_0}$ there is $y \in [M]^{\aleph_0} \cap M$ such that $x \subseteq y$.

$M \prec \mathcal{H}(\theta)$ is said to be *$\mathcal{H}(\kappa)$-like* if for any $x \in [M]^{<\kappa}$ there is $N \in M \cap [M]^{<\kappa}$ such that $x \subseteq N \prec M$. Clearly, if M is $\mathcal{H}(\aleph_1)$-like, then M is ω-bounding.

Since any internally approachable elementary submodel of $\mathcal{H}(\theta)$ of cardinality $\aleph_1$ is $\mathcal{H}(\aleph_1)$-like, there are cofinally many $\mathcal{H}(\aleph_1)$-like $M \prec \mathcal{H}(\theta)$ of cardinality $\aleph_1$. More generally, if $\mathrm{cf}([\kappa]^{\aleph_0}, \subseteq) = \kappa$ and θ is a sufficiently large regular cardinal, then there are cofinally many $\mathcal{H}(\omega_1)$-like elementary submodels $M \in [\mathcal{H}(\theta)]^{\kappa}$ of $\mathcal{H}(\theta)$.

Lemma 3.10. *Suppose that B is an $\aleph_2$-projective Boolean algebra. Then*

(1) *B satisfies the c.c.c.;*

(2) *B has the BSP; and*

(3) *B is ω-stable.*

Proof. (1): Suppose that B does not satisfy the c.c.c. and let $X \in [B^+]^{\aleph_1}$ be pairwise disjoint. By the $\aleph_2$-projectiveness of B, there is $A \in [B]^{\aleph_1}$ such that $A \leq B$, $X \subseteq A$ and A is projective. But since A satisfies the c.c.c. (see the remark after Theorem 1.3) this is a contradiction.

(2): Suppose that $I \subseteq B$ is a regular ideal on B. Let θ be a sufficiently large regular cardinal and let $M \prec \mathcal{H}(\theta)$ be ω-bounding with $|\,M\,| = \aleph_1$ and $B, I \in M$. Then $B \cap M$ is projective and hence has the BSP by Theorem 3.9. Since $I \cap M$ is a regular ideal in $B \cap M$, there is a countable $X \subseteq I \cap M$ generating $I \cap M$. Let $x \in M$ be such that x is countable and $X \subseteq x \subseteq I$. Then $M \models$ "x generates I". By elementarity x really generates I.

(3): Suppose that B were not ω-stable. Then there would be $X \subseteq [B]^{\aleph_0}$ and $b_\alpha \in B$, $\alpha < \omega_1$ such that $tp_X(b_\alpha)$, $\alpha < \omega_1$ are pairwise distinct. Let $A \in [B]^{\aleph_1}$ be such that $A \leq B$, A is projective and $X \cup \{b_\alpha : \alpha < \omega_1\} \subseteq A$. Then X and b_α, $\alpha < \omega_1$ witness that A is not ω-stable. This is a contradiction to Lemma 3.8. ❑ (Lemma 3.10)

Lemma 3.11. *Suppose that κ is a cardinal with $\mathrm{cf}([\kappa]^{\aleph_0}, \subseteq) = \kappa$ and θ is sufficiently large regular cardinal. Then there are cofinally many $M \in [\mathcal{H}(\theta)]^{\kappa}$ such that M is an $\mathcal{H}(\aleph_1)$-like elementary submodel of $\mathcal{H}(\theta)$.*

Proof. For an arbitrary $A \in [\mathcal{H}(\theta)]^{\kappa}$ we show that there is an $\mathcal{H}(\aleph_1)$-like elementary submodel M of $\mathcal{H}(\theta)$ such that $A \subseteq M$ and $|\,M\,| = \kappa$.

Let $\langle M_\alpha : \alpha < \omega_1 \rangle$, $\langle C_\alpha : \alpha < \omega_1 \rangle$ and $\langle D_\alpha : \alpha < \omega_1 \rangle$ be defined inductively such that

(3.1) $A \subseteq M_0$;

(3.2) $\langle M_\alpha : \alpha < \omega_1 \rangle$ is an increasing chain;

(3.3) $M_\alpha \prec \mathcal{H}(\theta)$ and $|M_\alpha| = \kappa$ for all $\alpha < \omega_1$;

(3.4) $C_\alpha \in [[M_\alpha]^{\aleph_0}]^\kappa$ and C_α is cofinal in $[M_\alpha]^{\aleph_0}$ with respect to $\subseteq$ for all $\alpha < \omega_1$;

(3.5) for all $\alpha < \omega_1$, we have $D_\alpha \in [[\mathcal{H}(\theta)]^{\aleph_0}]^\kappa$, $N \prec \mathcal{H}(\theta)$ for all $N \in D_\alpha$ and if $c \in C_\alpha$ then there is some $N \in D_\alpha$ with $c \subseteq N$;

(3.6) $D_\alpha \subseteq M_{\alpha+1}$ for all $\alpha < \omega_1$.

Let $M = \bigcup_{\alpha<\omega_1} M_\alpha$. Then this M is as desired: $A \subseteq M$ by (3.1) and $M \prec \mathcal{H}(\theta)$ by (3.2) and (3.3). Suppose that $c \in [M]^{\aleph_0}$. Then there is $\alpha < \omega_1$ such that $c \in [M_\alpha]^{\aleph_0}$. By (3.4) and (3.5), there is $N \in D_\alpha$ such that $c \subseteq N$. By (3.6), $N \in M_{\alpha+1} \subseteq M$. $N \prec M$ by (3.5). Thus M is $\mathcal{H}(\aleph_1)$-like.

❑ (Lemma 3.11)

Theorem 3.12. *Suppose that B is a c.c.c. ω-stable Boolean algebra with the BSP. Then, for a sufficiently large regular cardinal θ and ω-bounding $M \prec \mathcal{H}(\theta)$ with $B \in M$, we have $B \cap M \leq_{\mathrm{rc}} B$.*

Proof. Let B, θ and M be as above. For an arbitrary $b \in B$ we show that $q^{B\cap M}(b)$ exists.

Let U be a maximal pairwise disjoint subset of $B \cap M \restriction -b$. U is countable since B satisfies the c.c.c. Since M is ω-bounding, there is $S \in [M]^{\aleph_0} \cap M$ such that $U \subseteq S$. Let $T = \{tp_{B\cap S}(b) : b \in B\}$. Then $T \in M$. By ω-stability of B, $|T| = \aleph_0$. It follows that $T \subseteq M$. In particular, $tp_{B\cap S}(b) \in M$. Let $b' \in M$ be such that $tp_{B\cap S}(b') = tp_{B\cap S}(b)$.

Now, let $K = (B \cap S) \restriction -b = (B \cap S) \restriction -b'$. Then $K \in M$ and $U \subseteq K$. Let $J = K^\perp$ (where the operator $\perp$ acts with respect to the Boolean algebra B). J is a regular ideal on B and $J \in M$. Since B has the BSP, there is $X \in [J]^{\aleph_0}$ cofinal in J. By elementarity, we may assume that $X \in M$ and hence $X \subseteq M$. Since $b \in J$, there is some $d \in X$ such that $b \leq d$.

We claim that this d is the upper projection of b onto $B \cap M$. Suppose otherwise. Then there would be some $c \in B \cap M$ such that $b \leq c$ and $d \not\leq c$, i.e. $d \cdot -c \neq 0$. By the maximality of U, and since $d \cdot -c \leq -b$, there is some $e \in U$ such that $d \cdot -c \cdot e \neq 0$. But this is a contradiction to $d \in X \subseteq J = K^\perp \subseteq U^\perp$. ❑ (Theorem 3.12)

Theorem 3.13. *Suppose that* SSH *holds. Then every $\aleph_2$-projective Boolean algebras B have a filtration $\langle B_\alpha : \alpha < \kappa \rangle$ for $\kappa = \mathrm{cf}(|B|)$ such that $B_{\alpha+1}$*

is $\aleph_2$-projective and $B_{\alpha+1} \leq_\sigma B$ for all $\alpha < \kappa$. In particular, we also have $B_\alpha \leq_\sigma B$ for all limit $\alpha < \kappa$ of countable cofinality.

Proof. The assertion of the theorem is trivial if $|B| \leq \aleph_1$. So assume that $|B| \geq \aleph_2$. Let θ be a sufficiently large regular cardinal.

Case I. $|B| = \kappa = \lambda^+$ and $\mathrm{cf}(\lambda) > \omega$. By SSH, we have $\mathrm{cf}([\lambda]^{\aleph_0}, \subseteq) = \lambda$. Hence, by Lemma 3.11, there is an increasing chain $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels of $\mathcal{H}(\theta)$ such that

(3.7) $B \in M_0$;

(3.8) M_α is $\mathcal{H}(\omega_1)$-like for all $\alpha < \kappa$;

(3.9) $|M_\alpha| = \lambda$ for all $\alpha < \kappa$; and

(3.10) $B \subseteq \bigcup_{\alpha<\kappa} M_\alpha$.

Let $B_\alpha = B \cap (\bigcup_{\beta<\alpha} M_{\beta+1})$ for $\alpha < \kappa$. Then $\langle B_\alpha : \alpha < \kappa \rangle$ is a filtration of B.

If $\alpha < \kappa$ is 0 or a successor ordinal, then $B_\alpha = B \cap M_\alpha$. B_α is then $\aleph_2$-projective by Proposition 3.7. If α is a limit ordinal, since $M = \bigcup_{\beta<\alpha} M_{\beta+1}$ is an elementary submodel of $\mathcal{H}(\theta)$, $B_\alpha = B \cap M$ is also $\aleph_2$-projective by Proposition 3.7.

Thus the filtration $\langle B_\alpha : \alpha < \kappa \rangle$ is as desired.

Case II. $|B| > \kappa$. Similarly to Case I.

Case III. $|B| = \kappa = \lambda^+$ and $\mathrm{cf}(\lambda) = \omega$. Let $\lambda = \sup_{n\in\omega} \lambda_n$ where $\langle \lambda_n : n \in \omega \rangle$ is an increasing sequence of cardinals of cofinality $> \omega$.

For $\alpha < \kappa$ and $n < \omega$, let $M_{\alpha,n}$ be defined inductively such that

(3.11) $B \in M_{\alpha,0}$ for all $\alpha < \kappa$;

(3.12) $M_{\alpha,n} \prec \mathcal{H}(\theta)$ and $|M_{\alpha,n}| = \lambda_n$ for all $\alpha < \kappa$ and $n < \omega$;

(3.13) $M_{\alpha,n}$ is $\mathcal{H}(\aleph_1)$-like for all $\alpha < \kappa$ and $n < \omega$;

(3.14) $\langle M_{\alpha,n} : n \in \omega \rangle$ is an increasing chain for all $\alpha < \kappa$;

(3.15) $\langle M_{\alpha+1} : \alpha < \kappa \rangle$ is an increasing chain where

(3.15a) $M_{\alpha+1} = \bigcup_{n\in\omega} M_{\alpha,n}$

for $\alpha < \kappa$; and

(3.16) $B \subseteq \bigcup_{\alpha<\kappa} M_{\alpha+1}$.

The construction of $M_{\alpha,n}$'s is possible by SSH and Lemma 3.11.

Let

(3.17) $M_\gamma = \bigcup_{\alpha<\gamma} M_{\alpha+1}$ for all limit $\gamma < \kappa$

and let $B_\alpha = B \cap M_\alpha$ for all $\alpha < \kappa$. Then, by (3.12), (3.15), (3.16) and (3.17), we have $|\,B_\alpha\,| \leq \lambda$ for all $\alpha < \kappa$ and $\langle B_\alpha : \alpha < \kappa\rangle$ is a filtration of B. By (3.11), (3.13) and Theorem 3.12, we have $B \cap M_{\alpha,n} \leq_{\rm rc} B$ for all $\alpha < \kappa$ and $n < \omega$. It follows by the definition (3.15a) of $M_{\alpha+1}$ that $B_{\alpha+1} = B \cap M_{\alpha+1} = \bigcup_{n\in\omega} B \cap M_{\alpha,n} \leq_\sigma B$. By the definition of B_α's and Proposition 3.7, all of B_α, $\alpha < \kappa$ are $\aleph_2$-projective. ❑ (Theorem 3.13)

4 Openly generated Boolean algebras under FRP

In this section, we prove Theorem 1.6.

The implication "(a) ⇒ (b)" follows from Theorem 3.2, (d) and Lemma 3.1, (2).

The proof of the other implication "(b) ⇒ (a)" is done by induction on the cardinality of B.

For Boolean algebras of cardinality $\leq \aleph_1$, the implication clearly holds.

From now on we need the assumption of FRP. Suppose that we have shown the implication "(b) ⇒ (a)" for all Boolean algebras of cardinality $< \lambda$ for some cardinal $\lambda > \aleph_1$.

Case I: λ is regular. Let B be an $\aleph_2$-projective Boolean algebra of cardinality λ. Let $\mathcal{C} \subseteq \{C \in [B]^{\aleph_1} : C \text{ is projective}\}$ be closed unbounded in $[B]^{\aleph_1}$. By Theorem 3.13, there is a filtration $\langle B_\alpha : \alpha < \lambda\rangle$ of B such that all B_α, $\alpha < \lambda$ are $\aleph_2$-projective and $B_\alpha \leq_\sigma B$ for all $\alpha \in \lambda \setminus E^\kappa_{>\omega}$. Note that we may apply Theorem 3.13 here by Theorem 2.2. By the induction hypothesis, it follows that all B_α, $\alpha < \lambda$ are openly generated.

Suppose, toward a contradiction, that B were not openly generated.

Claim 4.0.1. *$E = \{\alpha \in E^\lambda_\omega : B_\alpha \leq_{\neg\rm rc} B\}$ is stationary in λ.*

⊢ Otherwise, there is a closed unbounded $C \subseteq \lambda$ such that, for every $\alpha \in C \cap E^\lambda_\omega$, we have $B_\alpha \leq_{\rm rc} B$. But then, by the c.c.c. of B, we can show that $B_\alpha \leq_{\rm rc} B$ for all $\alpha \in C \setminus E^\lambda_\omega$ as well and it follows that B is openly generated by Theorem 3.3. This is a contradiction to the assumption on B. ⊣ (Claim 4.0.1)

We may assume that (the underlying set of) B is λ. By thinning out the stationary set E, we may also assume that (the underlying set of) B_α is α for all $\alpha \in E$.

For each $\alpha \in E$, we have $B_\alpha \leq_\sigma B$. So let $\eta_\alpha \in B$ and $\eta^\alpha_n \in B_\alpha$, $n \in \omega$, be such that the ideal $B_\alpha \restriction \eta_\alpha$ is not generated by a single element but $\{\eta_n^\alpha : n \in \omega\}$ generates it. Let $g : E \to [\lambda]^{\aleph_0}$ be defined by $g(\alpha) = \{\eta_\alpha\} \cup \{\eta_n^\alpha : n \in \omega\}$ for $\alpha \in E$. By (the principle shown in Proposition 1.1 to be equivalent to) FRP, there is $I \in \mathcal{C}$ such that (1.1), (1.2) and (1.3) hold for I with the E and g as above.

Since $I \in \mathcal{C}$, I (as a subalgebra of B) is openly generated. On the other hand, by Lemma 1.2, I has a filtration $\langle I_\xi : \xi < \omega_1\rangle$ such that

$$S = \{\xi \in \omega_1 : \sup(I_\xi) \in I \text{ and } \{\eta_n^{\sup(I_\xi)} : n \in \omega\} \subseteq I_\xi\}$$

is stationary. Since $\eta_{\sup(I_\xi)} \in I$ for $\xi \in S \cap I$, it follows that $\{\xi \in \omega_1 : I_\xi \leq_{\neg\mathrm{rc}} I\} \supseteq S$ is also stationary. On the other hand, since I is openly generated, the filtration $\langle I_\xi : \xi < \omega_1\rangle$ has a continuous subsequence $\langle I'_\xi : \xi < \omega_1\rangle$ such that $I'_\xi \leq_{\mathrm{rc}} I$ for all $\xi < \omega_1$ by Theorem 3.2, (b). This is a contradiction.

Case II: λ is singular. Let B be an $\aleph_2$-projective Boolean algebra of cardinality λ and let $\mu = \mathrm{cf}(\lambda) < \lambda$. Without loss of generality, $\omega_1 \subseteq B$. Let $h : B^{<\omega} \to B$ be such that all $C \in [B]^{\aleph_1}$ closed with respect to h with $\omega_1 \subseteq C$ are projective subalgebras of B.

Let $\langle B_\alpha : \alpha < \mu\rangle$ be a filtration of B such that $\omega_1 \subseteq B_0$ and each B_α is closed with respect to h. Then each B_α is $\aleph_2$-projective and hence, by the induction hypothesis, openly generated. By Theorem 3.2, (c), there is an FN-mapping $f_\alpha : B_\alpha \to [B_\alpha]^{<\aleph_0}$ for each $\alpha < \mu$. Now let $\langle C_\xi : \xi < \mu\rangle$ be another filtration of B such that

(4.1) $\omega_1 \subseteq C_0$;

(4.2) C_ξ is closed with respect to h for all $\xi < \mu$;

(4.3) C_ξ is closed with respect to f_α, $\alpha < \mu$ for all $\xi < \mu$.

By (4.1), (4.2) and the induction hypothesis, all C_ξ, $\xi < \mu$ are openly generated.

For $\xi < \mu$ and $b \in B$ let $\alpha_0 < \mu$ be such that $b \in B_{\alpha_b}$. For $\alpha \in \mu \setminus \alpha_b$, let

(4.4) $b_\alpha^\xi = \sum^B \big(f_\alpha(b) \cap C_\xi \restriction b\big)$.

Note that b_α^ξ is well-defined since $f_\alpha(b)$ is finite.

Claim 4.0.2. *For $\xi < \mu$ and $b \in B$, $\langle b_\alpha^\xi : \alpha \in \mu \setminus \alpha_b\rangle$ is an increasing sequence cofinal in $C_\xi \restriction b$.*

$\vdash$ Suppose $\alpha_b \leq \alpha_0 < \alpha_1 < \mu$. Since $b_{\alpha_0}^\xi \leq b$, there is $c \in f_{\alpha_1}(b_{\alpha_0}^\xi) \cap f_{\alpha_1}(b)$ such that $b_{\alpha_0}^\xi \leq c \leq b$. Since $b_{\alpha_0}^\xi \in C_\xi$ by (4.4), we have $f_{\alpha_1}(b_{\alpha_0}^\xi) \subseteq C_\xi$ by

(4.3). Hence $c \in C_\xi \restriction b$. By (4.4) with $\alpha = \alpha_1$, it follows hat $c \le b^\xi_{\alpha_1}$. Thus $b^\xi_{\alpha_0} \le b^\xi_{\alpha_1}$.

Now, suppose that $c \in C_\xi \restriction b$. Let $\alpha^* \in \mu \setminus \alpha_b$ be such that $c \in B_{\alpha^*}$. Then there is $d \in f_{\alpha^*}(c) \cap f_{\alpha^*}(b)$ such that $c \le d \le b$. By (4.3) and since $c \in C_\xi$, we have $d \in C_\xi$. Thus, by (4.4), we have $c \le d \le b^\xi_{\alpha^*}$. This shows that $\{b^\xi_\alpha : \alpha \in \mu \setminus \alpha_0\}$ is cofinal in $C_\xi \restriction b$. $\dashv$ (Claim 4.0.2)

Case IIa: $\mu = \omega$. By Claim 4.0.2, $C_\xi \le_\sigma B$ for all $\xi < \mu$. By Bandlow's Theorem 3.4, it follows that B is openly generated.

Case IIb: $\mu > \omega$. In this case, we have the following:

Claim 4.0.3. *$C_\xi \le_{\rm rc} B$ for all $\xi < \mu$.*

$\vdash$ Otherwise, $\langle b^\xi_\alpha : \alpha \in \mu \setminus \alpha_b \rangle$ would be strictly increasing for some $b \in B$. Since B satisfies the c.c.c., this is a contradiction. $\dashv$ (Claim 4.0.3)

By Theorem 3.3, it follows that B is openly generated.

❑ (Theorem 1.6)

A Boolean algebra B is said to be $\mathcal{L}_{\infty,\aleph_2}$-projective if $B \models \psi$ holds for any $\mathcal{L}_{\infty,\aleph_2}$-sentence ψ which holds in all projective Boolean algebras. Similarly to [5] we obtain now under FRP the following:

Theorem 4.1. *Assume* FRP. *Then every $\mathcal{L}_{\infty,\aleph_2}$-projective Boolean algebra is openly generated.* ❑

In [5] a counterexample to the assertion of Theorem 4.1 is constructed under the existence of non-reflecting stationary set in E^κ_ω for some regular κ. This shows that the assertion of Theorem 4.1 above implies ORP.

Problem 1. *Does the assertion of* Theorem 4.1 *imply* FRP*?*

In the next section we show that the assertion of Theorem 1.6 implies and hence is equivalent to FRP.

5 Implication of FRP from the assertion of Theorem 1.6

As we already mentioned in Section 2, it is shown in [12, Theorem 2.5] that the negation of FRP is equivalent to the existence of a regular $\kappa > \aleph_1$ satisfying $\mathrm{ADS}^-(\kappa)$. That is, such κ that there is a stationary $S \subseteq E^\kappa_\omega$ and an almost essentially disjoint $g : S \to [\kappa]^{\aleph_0}$ with

(5.1) $g(\alpha) \subseteq \alpha$ and $\mathrm{otp}(g(\alpha)) = \omega$ for all $\alpha \in S$.

In [12, Lemma 2.3] it is shown that we may assume that g as above is a ladder system on S, that is, in addition to (5.1), we may also assume that $g(\alpha)$ is a cofinal subset of α for all $\alpha \in S$.

Proposition 5.1. *Suppose that $S \subseteq E^{\kappa}_{\omega}$ is a stationary set for a regular cardinal $\kappa \geq \aleph_2$ and $g : S \to [\kappa]^{\aleph_0}$ is an almost essentially disjoint ladder system. Then there is a Boolean algebra B of cardinality κ such that*

(5.2) *B is not openly generated but*

(5.3) *B is λ-openly generated for all regular $\lambda \leq \kappa$.*

Proof. Let S and g be as above. Without loss of generality, we may assume that $g(\alpha)$ consists of successor ordinals for all $\alpha \in S$.

Let $D = \{\alpha + 1 : \alpha < \kappa\}$ and let $X = \{c_\alpha : \alpha \in S \cup D\}$ where c_α, $\alpha \in S \cup D$ are pairwise distinct constant symbols. Let $<_B$ be the partial ordering on X defined by

(5.4) $c_\alpha <_B c_\beta$ if and only if $\alpha \in D$, $\beta \in S$ and $\alpha \in g(\beta)$

for c_α, $c_\beta \in X$.

Let B be the Boolean algebra generated from X freely except $<_B$. That is, $B = \mathrm{Fr}\, X/I_{<_B}$ where $I_{<_B}$ is the ideal on $\mathrm{Fr}\, X$ generated from $\{c_\alpha \cdot -c_\beta : \alpha, \beta \in S \cup D,\ c_\alpha <_B c_\beta\}$.

We show that this B satisfies (5.2) and (5.3). Note that elements of B can be represented uniquely by a term t in reduced disjunctive normal form built up from some elements of X. In the following we always identify such terms t with elements of B they represent. In particular, we consider X as a subset of B.

For $t \in B$, let

$$C(t) = \{c_\alpha : c_\alpha \text{ appears in } t\}.$$

Claim 5.1.1. $B \models$ (5.2), *i.e., B is not openly generated.*

$\vdash$ Otherwise there would be a FN-mapping $f : B \to [B]^{<\aleph_0}$. Let $f_0 : X \to [X]^{<\aleph_0}$ be defined by $f_0(c_\alpha) = \bigcup\{C(t) : t \in f(c_\alpha)\}$.

By Fodor's Lemma, there is a stationary $S' \subseteq S$ such that

$$f_0(c_\alpha) \cap \{c_\beta : \beta < \alpha\} \subseteq \{c_\beta : \beta < \delta^*\}$$

for some fixed $\delta^* < \kappa$ for all $\alpha \in S'$. By a further application of Fodor's Lemma, we obtain a stationary set $S'' \subseteq S'$ such that the first element in $g(\alpha)$ above δ^* is some fixed $\beta^* < \kappa$ for all $\alpha \in S''$. Let $\alpha^* \in S''$ be such that $f_0(c_{\beta^*}) \subseteq \{c_\beta : \beta < \alpha^*\}$. Since $\beta^* \in g(\alpha^*)$ we have $c_{\beta^*} <_B c_{\alpha^*}$. Then $f_0(c_{\beta^*}) \cap f_0(c_{\alpha^*}) \subseteq \delta^*$. It follows that $f(c_{\beta^*}) \cap f(c_{\alpha^*})$ cannot contain an element interpolating c_{α^*} and c_{β^*} with respect to the ordering $\leq_B$. This is a contradiction. $\dashv$ (Claim 5.1.1)

Claim 5.1.2. $B \models (5.3)$. *More specifically, for any $Y \in [X]^{<\kappa}$, $[Y]_B$ is openly generated.*

$\vdash$ Suppose $Y \in [X]^{<\kappa}$. Let $\bar{Y} = \{\alpha \in S \cup D : c_\alpha \in Y\}$. Since g is almost essentially disjoint, there is a regressive $h : S \cap \bar{Y} \to \kappa$ such that $g(\alpha) \setminus h(\alpha)$, $\alpha \in S \cap \bar{Y}$ are pairwise disjoint. Let $f_0 : Y \to [Y]^{<\aleph_0}$ be defined by

$$f_0(c_\alpha) = \begin{cases} \{c_\alpha, c_\beta\} & \text{if } \beta \in S \cap \bar{Y} \text{ and} \\ & \alpha \in g(\beta) \setminus h(\beta), \\ \{c_\alpha\} \cup (\{c_\beta : \beta \in g(\alpha) \cap h(\alpha)\} \cap Y) & \text{if } \alpha \in S \cap \bar{Y}, \\ \{c_\alpha\} & \text{otherwise} \end{cases}$$

for $c_\alpha \in Y$.

Clearly f_0 is an FN-mapping (for the partial ordering $\langle Y, <_B \restriction Y \rangle$). Now let $f : [Y]_B \to [[Y]_B]^{<\aleph_0}$ be defined by

$$f(t) = [\,\bigcup\{f_0(c) : c \in C(t)\} \cup C(t)\,]_B$$

for $t \in [Y]_B$. Then f is an FN-mapping on $[Y]_B$. $\dashv$ (Claim 5.1.2)

❑ (Proposition 5.1)

Theorem 5.2. *The assertion of* Theorem 1.6 *is equivalent to* FRP *over* ZFC*.*

Proof. By Theorem 1.6, Proposition 5.1 and by [12, Theorem 2.5].

❑ (Theorem 5.2)

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