Fodor-type Reflection Principle, metrizability and meta-Lindelöfness

Sakaé Fuchino, István Juhász, Lajos Soukup,
Zoltán Szentmiklóssy and Toshimichi Usuba

Abstract

We introduce a new reflection principle which we call *"Fodor-type Reflection Principle"* (FRP). This principle follows from but strictly weaker than Fleissner's Axiom R. So, for example, FRP does not impose any restriction on the size of the continuum, while Axiom R implies that the continuum has size $\leq \aleph_2$.

We show that FRP implies that every locally separable countably tight topological space X is meta-Lindelöf if all of its subspaces of cardinality $\leq \aleph_1$ are meta-Lindelöf (Theorem 4.1). It follows from this theorem that, under FRP, every locally countably compact space X is metrizable if all of its subspaces of cardinality $\leq \aleph_1$ are metrizable (Corollary 4.4). This improves a result of Balogh who proved the same assertion under Axiom R.

1 Introduction

In this note, we consider the following type of reflection property of a topological space X. Let $\mathcal{P}$ be a property of a topological space and κ a cardinal.

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(1.1) If a topological space X satisfies the property $\mathcal{P}$, then there is a subspace of X of size $< \kappa$ satisfying the property $\mathcal{P}$.

By moving to the negation $\mathcal{Q}$ of $\mathcal{P}$, (1.1) can be also seen as the transfer property of the from:

(1.2) If every subspace of X of size $< \kappa$ satisfies the property $\mathcal{Q}$, then X also satisfies $\mathcal{Q}$.

The instance of (1.2), where κ is $\aleph_2$ and $\mathcal{Q}$ "*metrizable*", is studied extensively in the literature with the most prominent result in this context being the following theorem of Dow.

Definition 1.1. A topological space X is called $\aleph_1$-*metrizable* if every subspace of X of size $\leq \aleph_1$ is metrizable. More generally, X is said to be κ-*metrizable* ($< \kappa$-*metrizable* resp.) for a cardinal κ if every subspace of X of size $\leq \kappa$ ($< \kappa$ resp.) is metrizable.

Theorem 1.1. (A. Dow [6, Theorem 3.1]) *Every countably compact $\aleph_1$-metrizable space X is metrizable.* ❑

In particular, every compact $\aleph_1$-metrizable space is metrizable. Arhangelskii [1] asked if every locally compact $\aleph_1$-metrizable space is metrizable. Balogh proved that the answer is positive under Fleissner's Axiom R:

Theorem 1.2. (Z. Balogh [3, Theorem 2.2]) *Assume Axiom R. Then every locally compact $\aleph_1$-metrizable space is metrizable.* ❑

Recall that *Axiom R* is the principle asserting that the following $\mathrm{AR}([\kappa]^{\aleph_0})$ holds for all cardinals $\kappa \geq \aleph_2$:

$\mathrm{AR}([\kappa]^{\aleph_0})$: For any stationary $S \subseteq [\kappa]^{\aleph_0}$ and ω_1-club $T \subseteq [\kappa]^{\aleph_1}$, there is $I \in T$ such that $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Here, $T \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-*club* (or *tight and unbounded* in Fleissner's terminology in Fleissner [9]) if

(1.3) T is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and

(1.4) for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in T of length ω_1, we have $\bigcup_{\alpha<\omega_1} I_\alpha \in T$.

The assumption of Axiom R cannot be simply dropped from Theorem 1.2 since, under the existence of a non-reflecting stationary set, we obtain a very strong form of negative answer to the question of Arhangelskii as the next proposition shows. However, we shall prove in Section 4 that we can still weaken the assumption of Axiom R in the theorem to Fodor-type Reflection Principle which will be defined in Section 2 (see Definition 2.1) and that this principle is strictly weaker than Axiom R (see Section 3).

Given a topological space X and a family $\mathcal{F}$ of open sets, let $\mathrm{ord}(x,\mathcal{F}) = |\{F \in \mathcal{F} : x \in F\}|$ for $x \in X$ and $\mathrm{ord}(\mathcal{F}) = \sup\{\mathrm{ord}(x,\mathcal{F}) : x \in X\}$. We say that $\mathcal{F}$ is *point countable* if $\mathrm{ord}(\mathcal{F}) \leq \aleph_0$.

Recall that a topological space X is said to be *meta-Lindelöf* if every open cover $\mathcal{B}$ of X has a point countable refinement. It is clear that every paracompact space is meta-Lindelöf. By the Stone theorem, every metrizable space is paracompact.

Proposition 1.3. *If there is a non-reflecting stationary set $S \subseteq E^{\kappa}_{\omega}$ for a regular cardinal $\kappa \geq \aleph_2$ then there is a non-meta-Lindelöf (and hence non-metrizable), locally compact, locally countable $<\kappa$-metrizable space X of size κ.*

Note that the usual subspace topology on such an S is non-metrizable and $\aleph_1$-metrizable, but not locally compact.

Proof. Let $I = \{\xi + 1 : \xi < \kappa\}$. The underlying set of X is the disjoint union $S \cup I$. For each $\alpha \in S$ choose a countable subset $a_\alpha \in [I \cap \alpha]^{\aleph_0}$ of order type ω which is cofinal in α. Now define the topology of X as follows:

(1.5) the elements of I are isolated.

(1.6) a neighborhood base of $\alpha \in S$ is $\{\{\alpha\} \cup (a_\alpha \setminus \beta) : \beta < \alpha\}$.

By the Fodor lemma, every open cover $\mathcal{B}$ of X has a point in X which is covered by κ many elements of $\mathcal{B}$. It follows that X is not meta-Lindelöf and, in particular, non-metrizable.

X is clearly locally compact, so it is also regular. By the definition of the topology on X, it is also clear that X is locally countable.

We show, by induction on δ, that, for all $\delta < \kappa$, $X \restriction \delta$ has a σ-discrete base. Since κ is regular, this is enough by the Bing metrization theorem.

The only non-trivial step is when $\mathrm{cf}(\delta) \geq \omega_1$. But then there is a club $C \subseteq \delta$ with $C \cap (S \cap \delta) = \emptyset$. Let γ_ν, $\nu < \lambda$ be an increasing enumeration of C. Then $X \restriction \delta$ has the partition $\{X \restriction [\gamma_\nu, \gamma_{\nu+1}) : \nu < \lambda\}$ into clopen sets.

Each of the clopen sets has σ-discrete base by the induction hypothesis. So $X \restriction \delta$ also does. ❑ (Proposition 1.3)

In Section 2, we introduce a new type of stationary reflection principle FRP (or FRP(κ) for regular cardinals $\kappa \geq \aleph_1$) to which the role of Axiom R in Balogh's theorem can be localized. We show that FRP(κ) follows from RP($[\kappa]^{\aleph_0}$) (Theorem 2.5) where RP($[\kappa]^{\aleph_0}$) is a slight strengthening of what is called "Reflection Principle" and denoted by RP(κ) in Jech's Millennium Book [13]. Since Axiom R implies RP($[\kappa]^{\aleph_0}$) for all cardinals κ of cofinality $\geq \omega_1$, FRP is a consequence of Axiom R.

On the other hand, we show in Section 3 that it is consistent that FRP(κ) for all cardinal κ of cofinality $\geq \omega_1$ holds while RP($[\kappa]^{\aleph_0}$) does not hold for all $\kappa \geq \aleph_2$ (Theorem 3.5, (1)).

In Section 4, we prove that the transfer property (1.2) holds for meta-Lindelöfness of locally compact spaces under FRP (Theorem 4.1). We show that the assertion of Balogh's theorem (Theorem 1.2) follows from Theorem 4.1 (Corollary 4.4). Since FRP is strictly weaker than Axiom R, Corollary 4.4 is an essential improvement of Balogh's theorem. Also, Theorem 3.5, (2) implies that these topological transfer theorems under FRP do not impose any restriction on the size of the continuum.

Since FRP(ω_1) is simply equivalent to the Fodor lemma, we can easily single out the ZFC part of the proofs of these transfer theorems to obtain the corresponding ZFC results (Corollaries 4.5 and 4.6).

For a property $\mathcal{Q}$, let us call a topological space *almost* $\mathcal{Q}$ if every subspace Y of X of cardinality $< |X|$ satisfies $\mathcal{Q}$. In particular, X is almost metrizable if and only if X is $<|X|$-metrizable.

A natural variant of (1.2) would be:

(1.7) If X is almost $\mathcal{Q}$, then X satisfies $\mathcal{Q}$.

For various properties $\mathcal{Q}$, we can ask whether (1.7) holds for all topological spaces X in a given class $\mathcal{C}$ of topological spaces and consider this problem as a question on compactness of $\mathcal{C}$ (in the sense of abstract model theory) with respect to the property $\mathcal{Q}$.

In Section 5, we present miscellaneous results concerning the metrizability (resp. meta-Lindelöfness) of almost metrizable (resp. almost meta-Lindelöf) spaces X in various classes $\mathcal{C}$ of topological spaces.

Proposition 1.3 can be also seen as an anticompactness result:

Proposition 1.4 (A rephrasing of Proposition 1.3)**.** *If there is a non-reflecting stationary set* $S \subseteq E^{\kappa}_{\omega}$ *for a regular cardinal* $\kappa \geq \aleph_2$ *then there is a locally compact, locally countable space* X *of size* κ *which is almost metrizable but not meta-Lindelöf (and hence not metrizable).*

In Section 6, we show that the same kind of anticompactness of metrizability as Proposition 1.4 can also hold without the existence of non-reflecting stationary sets.

We tried to make this note easily accessible for both topologists and set-theorists, though some of the readers good in both fields might find our formulation a little bit overdetailed.

2 Fodor-type Reflection Principle

In this section, we introduce the principle which we call "Fodor-type Reflection Principle (FRP)" and show that this principle follows from Axiom R. We show in the next section that this principle is strictly weaker than Axiom R and some other weakenings of it.

The applications of FRP on reflection properties of topological spaces mentioned in the introduction will be given in Section 4. Actually, it appears that most of the known applications of Axiom R are already provable under FRP (see also Fuchino, Sakai, Soukup and Usuba [12]).

Definition 2.1. Let κ be a cardinal of cofinality $\geq \omega_1$. *The Fodor-type Reflection Principle for* κ (FRP(κ)) is the following statement:

FRP(κ): For any stationary $S \subseteq E^{\kappa}_{\omega}$ and mapping $g : S \to [\kappa]^{\leq\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

(2.1) $\mathrm{cf}(I) = \omega_1$;

(2.2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;

(2.3) for any regressive $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

Note that, for S and I as above, $S \cap I$ is stationary in $\sup(I)$. In particular, if $S \cap I$ were empty, then $\emptyset : S \cap I \to \kappa$ is a/the regressive function for which there is no ξ^* as in (2.3).

Fact 2.1. FRP(ω_1) *holds in* ZFC.

Indeed, if we take $I = \omega_1$ then the statement follows immediately from the Fodor Lemma.

Lemma 2.2. FRP(κ) *fails for a singular* κ.

Proof. Suppose that $\lambda = \mathrm{cf}(\kappa) < \kappa$. Let $\langle \alpha_\xi : \xi < \lambda \rangle$ be a continuously and strictly increasing sequence of ordinals in $\kappa \setminus \lambda$ cofinal in κ. Let $S = \{\alpha_\xi : \xi \in E^\lambda_\omega\}$. Then $S \subseteq E^\kappa_\omega$ and S is stationary in κ. Let $g : S \to \kappa$ be defined by

$$g(\alpha_\xi) = \{\xi\} \text{ for } \xi \in E^\kappa_\omega. \tag{2.4}$$

Since $S \ni \alpha_\xi \mapsto \xi \in \lambda$ is regressive and strictly increasing, there cannot be any $I \in [\kappa]^{\aleph_1}$ satisfying (2.3). This shows that FRP(κ) does not hold.

❑ (Lemma 2.2)

Definition 2.2. Let FRP be the asssertion: FRP(κ) holds for all regular $\kappa \geq \aleph_1$.

For regular $\kappa \geq \aleph_2$, FRP(κ) is not provable in ZFC since, for example, the existence of a non-reflecting subset of E^κ_ω would refute FRP(κ). In section 6, we show that the non existence of non-reflecting subset of E^κ_ω even does not guarantee FRP(κ).

However we can show that FRP(κ) follows from RP($[\kappa]^{\aleph_0}$) (see Theorem 2.5).

Here, for a cardinal $\kappa \geq \aleph_2$, RP($[\kappa]^{\aleph_0}$) is the following principle:

RP($[\kappa]^{\aleph_0}$): For any stationary $S \subseteq [\kappa]^{\aleph_0}$, there is an $I \in [\kappa]^{\aleph_1}$ such that

$$\omega_1 \subseteq I; \tag{2.5}$$

$$\mathrm{cf}(I) = \omega_1; \tag{2.6}$$

$$S \cap [I]^{\aleph_0} \text{ is stationary in } [I]^{\aleph_0}. \tag{2.7}$$

The following is well-known:

Lemma 2.3. RP($[\kappa]^{\aleph_0}$) *is equivalent to the assertion that for any stationary* $S \subseteq [\kappa]^{\aleph_0}$, *there are stationarily many* $I \in [\kappa]^{\aleph_1}$ *satisfying* (2.5), (2.6) *and* (2.7). ❑

AR($[\kappa]^{\aleph_0}$) implies RP($[\kappa]^{\aleph_0}$) for a cardinal κ of cofinality $\geq \omega_1$ since $T = \{I \in [\kappa]^{\aleph_0} : \omega_1 \subseteq I \text{ and } \mathrm{cf}(I) = \omega_1\}$ is ω_1-club. Jech [13] called a weakening of RP($[\kappa]^{\aleph_0}$) "*Reflection Principle*" which is obtained by dropping the condition (2.6) from the definition of RP($[\kappa]^{\aleph_0}$). Jech's reflection principle is sometimes also called "*Weak Reflection Principle*" in the literature (see, e.g. König, Larson and Yoshinobu [14]) and so we denote this principle by $\mathrm{WRP}([\kappa]^{\aleph_0})$.

Axiom R follows from $\mathrm{MA}^+(\sigma\text{-closed})$ (see Beaudoin [4]) which in turn is a consequence of Martin's Maximum (see Foreman, Magidor and Shelah [10]). In more modern terminology of Foreman and Todorcevic [11], Axiom R is equivalent to the stationary reflection to a internally unbounded structure (this fact is stated essentially in Dow [7] under the definition of Axiom R which is slightly stronger than the one we use here). Since $\mathrm{MA}^+(\sigma\text{-closed})$ is consistent with CH (modulo some large cardinal), all the reflection principles we treat here are compatible with CH.

It is still open if $\mathrm{WRP}([\kappa]^{\aleph_0})$, $\mathrm{RP}([\kappa]^{\aleph_0})$ and $\mathrm{AR}([\kappa]^{\aleph_0})$ can be separated. This seems to be a quite difficult problem if these principles should be ever separated: it is known that $\mathrm{RP}([\omega_2]^{\aleph_0})$ and $\mathrm{AR}([\omega_2]^{\aleph_0})$ are equivalent; under $2^{\aleph_1} = \aleph_2$, $\mathrm{WRP}([\omega_2]^{\aleph_0})$ and $\mathrm{RP}([\omega_2]^{\aleph_0})$ are equivalent and, e.g. under GCH, $\mathrm{WRP}([\omega_n]^{\aleph_0})$ and $\mathrm{RP}([\omega_n]^{\aleph_0})$ for all $n \in \omega$ are equivalent (see König, Larson and Yoshinobu [14]).

Nevertheless, our Fodor-type Reflection Principle can be easily separated from these reflection principles (see the next section).

Let us begin with a useful characterization of $\mathrm{FRP}(\kappa)$:

Lemma 2.4. *For a regular cardinal* $\kappa \geq \aleph_2$*,* $\mathrm{FRP}(\kappa)$ *is equivalent to the following* $\mathrm{FRP}^\bullet(\kappa)$*:*

$\mathrm{FRP}^\bullet(\kappa)$ *: For any stationary* $S \subseteq E^\kappa_\omega$ *and mapping* $g : S \to [\kappa]^{\leq\aleph_0}$ *there is a continuously increasing sequence* $\langle I_\xi : \xi < \omega\rangle$ *of countable subsets of* κ *such that*

(2.8) $\langle \sup(I_\xi) : \xi < \omega_1\rangle$ *is strictly increasing;*

(2.9) *each* I_ξ *is closed with respect to* g *and*

(2.10) $\{\xi < \omega_1 : \sup(I_\xi) \in S$ *and* $g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$ *is stationary in* ω_1*.*

Proof. First, assume $\mathrm{FRP}(\kappa)$. Let $S \subseteq E^\kappa_\omega$ be stationary and $g : S \to [\kappa]^{\aleph_0}$. Without loss of generality, we may assume that $g(\alpha) \cap \alpha \neq \emptyset$ for all $\alpha \in S$.

Let $I \in [\kappa]^{\aleph_1}$ be as in the definition of $\mathrm{FRP}(\kappa)$ for these S and g, and let $\langle I_\xi : \xi < \omega_1\rangle$ be a filtration of I satisfying (2.8) and (2.9). This is possible by (2.1) and (2.2).

We show that $\langle I_\xi : \xi < \omega_1 \rangle$ satisfies (2.10) as well. Suppose not. Then $\{\xi < \omega_1 : \sup(I_\xi) \notin S$ or $g(\sup(I_\xi)) \cap \sup(I_\xi) \not\subseteq I_\xi\}$ includes a club set $\subseteq \omega_1$. It follows that $S \cap I \setminus S_0$ is non stationary in $\sup(I)$ for

$$S_0 = \{\alpha \in S \cap I : \alpha = \sup(I_\xi) \text{ for some } \xi < \omega_1 \text{ and } g(\alpha) \cap \alpha \not\subseteq I_\xi\}.$$

Let $f : S \cap I \to I$ be defined by

$$f(\alpha) = \begin{cases} \min(g(\alpha) \cap \alpha \setminus I_\xi) & \text{if } \alpha \in S_0 \text{ and } \alpha = \sup(I_\xi); \\ \min(g(\alpha)) & \text{otherwise.} \end{cases}$$

Then f is regressive and $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$. By the assumption, there is an $\alpha^* \in I$ such that $f^{-1}{}''\{\alpha^*\}$ is stationary. In particular, $S_0 \cap f^{-1}{}''\{\alpha^*\}$ is stationary. Let $\xi^* \in \omega_1$ be such that $\alpha^* \in I_{\xi^*}$ and let $\beta \in S_0 \cap f^{-1}{}''\{\alpha^*\}$ be such that $\beta > \sup(I_{\xi^*})$. Let $\eta < \omega_1$ be such that $\beta = \sup(I_\eta)$. Then $\alpha^* \in I_{\xi^*} \subseteq I_\eta$. Since $\beta \in S_0$, we have $f(\beta) \notin I_\eta$ by the definition of f. It follows that $f(\beta) \neq \alpha^*$. This is a contradiction.

Now, assume $\mathrm{FRP}^\bullet(\kappa)$. Suppose that $S \subseteq E^\kappa_\omega$ is stationary and $g : S \to [\kappa]^{\aleph_0}$. Let $S_0 = \{\alpha \in S : \alpha$ is closed with respect to $g\}$. Since κ is regular S_0 is still stationary. Let $\langle I_\xi : \xi < \omega_1 \rangle$ be as in the definition of $\mathrm{FRP}^\bullet(\kappa)$ for S_0 and $g \restriction S_0$. Let I be the closure of $\bigcup_{\xi<\omega_1} I_\xi \cup \{\sup(I_\xi) : \xi < \omega_1\}$ with respect to g. By the definition of S_0 and since $\sup(I_\xi) \in S_0$ for stationary many $\xi < \omega_1$, we have $\sup(I) = \sup(\bigcup_{\xi<\omega_1} I_\xi)$. In particular, $\{\sup I_\xi : \xi < \omega_1\}$ is a club subset of $\sup(I)$.

We claim that this I satisfies the conditions in the definition of $\mathrm{FRP}(\kappa)$. It is clear that I satisfies (2.1) and (2.2). To see that it also satisfies (2.3), suppose that $f : S \cap I \to \kappa$ is regressive and $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$. By the assumption, $\{\xi \in \omega_1 : g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$ is stationary. Hence, letting $S_1 = \{\xi \in \omega_1 : f(\sup(I_\xi)) \in I_\xi\}$, we have that

$$S_1 \supseteq \{\xi \in \omega_1 : g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$$

is also stationary. For each $\xi \in S_1$, let

$$h(\xi) = \min\{\eta < \omega_1 : f(\sup(I_\xi)) \in I_\eta\}.$$

Then the mapping $h : S_1 \to \omega_1$ is regressive. Thus, by the Fodor lemma, there is a stationary $S_2 \subseteq S_1$ such that $h''S_2 = \{\eta^*\}$ for some $\eta^* \in \omega_1$. Since I_{η^*} is countable, there is a stationary $S_3 \subseteq S_2$ such that, for any $\xi \in S_3$, $f(\sup(I_\xi)) = \alpha^*$ for some fixed $\alpha^* \in I_{\eta^*}$. It follows that $f^{-1}{}''\{\alpha^*\} \supseteq \{\sup(I_\xi) : \xi \in S_3\}$ is stationary in $\sup(I)$. ❑ (Lemma 2.4)

Theorem 2.5. *For any regular cardinal $\kappa > \aleph_1$,* $\mathrm{RP}([\kappa]^{\aleph_0})$ *implies* $\mathrm{FRP}(\kappa)$*.*

Proof. By Lemma 2.4, it is enough to show that $\mathrm{RP}([\kappa]^{\aleph_0})$ implies $\mathrm{FRP}^{\bullet}(\kappa)$. Suppose that $S \subseteq E^{\kappa}_{\omega}$ is stationary and $g : S \to [\kappa]^{\leq\aleph_0}$. Let

(2.11) $S_0 = \{a \in [\kappa]^{\aleph_0} \,:\, \sup(a) \in S \setminus a,\ g(\sup(a)) \cap \sup(a) \subseteq a\}$.

Claim 2.5.1. *S_0 is a stationary subset of $[\kappa]^{\aleph_0}$.*

$\vdash$ Suppose that $C \subseteq [\kappa]^{\aleph_0}$ is a club. Let $s : \kappa^{<\omega} \to \kappa$ be such that $C \supseteq C(s) = \{a \in [\kappa]^{\aleph_0} \,:\, s''a^{<\omega} \subseteq a\}$ and let $D = \{\alpha < \kappa \,:\, s''\alpha^{<\omega} \subseteq \alpha\}$. Since κ is regular, D is a club subset of κ. So there is an $\alpha^* \in S \cap D$. Let $\langle \alpha_n \,:\, n \in \omega\rangle$ be an increasing sequence of ordinals such that $\alpha^* = \sup_{n\in\omega} \alpha_n$. Let a^* be the closure of $a_0 = \{\alpha_n \,:\, n \in \omega\} \cup (g(\alpha^*) \cap \alpha^*)$ with respect to s. Since a_0 is cofinal in α^* and $\alpha^* \in D$, we have $\sup(a^*) = \alpha^*$. Thus $a^* \in S_0$. By the closedness of a^* with respect to s, we also have $a^* \in C(s) \subseteq C$. $\dashv$ (Claim 2.5.1)

By $\mathrm{RP}([\kappa]^{\aleph_0})$, there is $I \in [\kappa]^{\aleph_1}$ such that

(2.12) $\mathrm{cf}(I) = \omega_1$;

(2.13) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;

(2.14) $S_0 \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Note that (2.13) is possible by Lemma 2.3.

Let $\langle I_\xi \,:\, \xi < \omega_1\rangle$ be a continuously increasing sequence of countable sets with $I = \bigcup_{\xi<\omega_1} I_\xi$ such that $\langle \sup(I_\xi) \,:\, \xi < \omega_1\rangle$ is strictly increasing (this is possible by (2.12)).

Let

$$S_1 = \{\xi < \omega_1 \,:\, \xi \text{ is a limit and } I_\xi \in S_0\} \text{ and}$$
$$S_2 = \{\xi < \omega_1 \,:\, g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi)\}.$$

By (2.14), S_1 is a stationary subset of ω_1. Now, by the definition (2.11) of S_0, we have $S_2 \supseteq S_1$. Thus S_2 is stationary as well. ❑ (Theorem 2.5)

Corollary 2.6. RP *implies* FRP*. In particular,* Axiom R *implies* FRP*.* ❑

3 Separation of FRP from WRP

In this section, we prove the consistency of the Fodor-type Reflection Principle with the total negation of the Weak Reflection Principle.

The following lemma is well-known and easy to prove:

Lemma 3.1. *For $\aleph_2 \leq \kappa \leq \kappa'$, if* $\mathrm{WRP}([\kappa']^{\aleph_0})$ *then* $\mathrm{WRP}([\kappa]^{\aleph_0})$. ❑

For a proof of the following lemma, see e.g. Jech [13], Theorem 37.18.

Lemma 3.2 (S. Todorčević)**.** $\mathrm{WRP}([\aleph_2]^{\aleph_0})$ *implies* $2^{\aleph_0} \leq \aleph_2$. ❑

Lemma 3.3 (S. Shelah)**.** *Suppose that $\mathbb{P}$ is a c.c.c. poset, S a stationary subset of ω_1 and $p_\alpha \in \mathbb{P}$ for $\alpha \in S$. Then, for all but countably many $\beta \in S$, we have*

$$p_\beta \Vdash_{\mathbb{P}} \text{“}\{\alpha \in S \,:\, p_\alpha \in \dot{G}\} \textit{ is stationary in } \omega_1\text{”}.$$

Proof. Suppose otherwise. Then we can construct a strictly increasing sequence $\langle \alpha_\xi \,:\, \xi < \omega_1 \rangle$ in S such that, for all $\xi < \omega_1$, there are $q_\xi \leq_{\mathbb{P}} p_{\alpha_\xi}$ and club $C_\xi \subseteq \omega_1$ such that

$$(3.1) \quad q_\xi \Vdash_{\mathbb{P}} \text{“}\{\alpha \in S \,:\, p_\alpha \in \dot{G}\} \cap C_\xi = \emptyset\text{”}.$$

Let $\beta_\xi \in \omega_1$, $\xi < \omega_1$ be such that

$$(3.2) \quad \beta_\xi \in S \cap \bigcap \{C_\eta \,:\, \eta < \xi\}.$$

Claim 3.3.1. $\{q_{\beta_\xi} \,:\, \xi < \omega_1\}$ *is an antichain.*

$\vdash$ For $\xi < \xi' < \omega_1$, we have $\beta_{\xi'} \in C_\xi$ by (3.2). Hence $q_{\beta_\xi} \Vdash_{\mathbb{P}} \text{“}p_{\beta_{\xi'}} \notin \dot{G}\text{”}$ by (3.1). Since $q_{\beta_{\xi'}} \leq_{\mathbb{P}} p_{\beta_{\xi'}}$, it follows that $q_{\beta_\xi} \Vdash_{\mathbb{P}} \text{“}q_{\beta_{\xi'}} \notin \dot{G}\text{”}$. Hence q_{β_ξ} and $q_{\beta_{\xi'}}$ are incompatible. $\dashv$ (Claim 3.3.1)

Now, by the c.c.c. of $\mathbb{P}$, this is a contradiction. ❑ (Lemma 3.3)

Theorem 3.4. *Suppose that* $\mathrm{FRP}(\kappa)$ *holds and $\mathbb{P}$ is a c.c.c. poset. Then* $\Vdash_{\mathbb{P}}$ “$\mathrm{FRP}(\kappa)$ *holds*”.

Proof. Suppose that $\dot{S}$ is a $\mathbb{P}$-name of a stationary subset of κ and $\dot{g}$ a $\mathbb{P}$-name of a mapping from $\dot{S}$ to $[\kappa]^{\aleph_0}$. Let

$$(3.3) \quad S = \{\alpha \in \kappa \,:\, p \Vdash_{\mathbb{P}} \text{“}\alpha \in \dot{S} \text{ for some } p \in \mathbb{P}\text{”}\}.$$

Then S is a stationary subset of κ. Let $g : S \to [\kappa]^{\aleph_0}$ be defined by

$$(3.4) \quad g(\alpha) = \{\beta \in \kappa \,:\, p \Vdash_{\mathbb{P}} \text{“}\beta \in \dot{g}(\alpha) \text{ for some } p \in \mathbb{P}\text{”}\}$$

for $\alpha \in S$. g is well-defined by the c.c.c. of $\mathbb{P}$.

By Lemma 2.4, there is a continuously increasing sequence $\langle I_\xi \,:\, \xi < \omega_1 \rangle$ with $I_\xi \in [\kappa]^{\aleph_0}$ for $\xi < \omega_1$ such that

(3.5) $\langle \sup(I_\xi) : \xi < \omega_1 \rangle$ is strictly increasing,

(3.6) I_ξ is closed with respect to g for all $\xi < \omega_1$, and

(3.7) $S_1 = \{\xi \in \omega_1 : g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$ is stationary.

For $\xi \in S_1$, since $\sup(I_\xi) \in S$, there is a $p_\xi \in \mathbb{P}$ such that $p_\xi \Vdash_{\mathbb{P}}$ " $\sup(I_\xi) \in \dot{S}$ ". Hence, by Lemma 3.3, there is a $\xi^* \in S_1$ such that

$$p_{\xi^*} \Vdash_{\mathbb{P}} \text{“}\{\xi \in S_1 : p_\xi \in \dot{G}\} \text{ is stationary in } \omega_1\text{”}.$$

Let $\dot{S}_2$ be a $\mathbb{P}$-name of "$\{\xi \in S_1 : p_\xi \in \dot{G}\}$". By the definition (3.4) of g

(3.8) $\Vdash_{\mathbb{P}}$ "$\dot{g}(\alpha) \subseteq g(\alpha)$ for every $\alpha \in \dot{S}$".

So, by the definition (3.7) of S_1, we have

$$p_{\xi^*} \Vdash_{\mathbb{P}} \text{“}\dot{S}_2 \subseteq \{\xi \in \omega_1 : \dot{g}(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}\text{”}.$$

By (3.8) and (3.6),

$$\Vdash_{\mathbb{P}} \text{“each } I_\xi,\ \xi < \omega_1, \text{ is closed with respect to } \dot{g}\text{”}.$$

Thus p_{ξ^*} forces that $\langle I_\xi : \xi < \omega_1 \rangle$ is as in the definition of $\mathrm{FRP}^\bullet(\kappa)$ for $\dot{S}$ and $\dot{g}$.

Since the argument above can be repeated in $\mathbb{P} \restriction p$ for any $p \in \mathbb{P}$, it follows that $\Vdash_{\mathbb{P}}$ "$\mathrm{FRP}(\kappa)$". ❑ (Theorem 3.4)

Theorem 3.5. (1) *Suppose that* "ZFC + FRP" *is consistent. Then so is* "ZFC + FRP + $\neg\mathrm{WRP}([\kappa]^{\aleph_0})$ *for all* $\kappa \geq \aleph_2$".

(2) *If* "ZFC + CH + FRP" *is consistent, then* "ZFC + FRP" *is consistent with any value of the size of continuum possible under* ZFC.

Proof. (1): Suppose that $V \models$ "ZFC + FRP". In V, let $\mathbb{P} = \mathbb{C}_\lambda$ (= the Cohen forcing adding λ many Cohen reals) for some $\lambda \geq \aleph_3$. Then $V^{\mathbb{P}} \models 2^{\aleph_0} \geq \aleph_3$. Hence, by Lemma 3.2 and Lemma 3.1, $V^{\mathbb{P}} \models$ "$\neg\mathrm{WRP}([\kappa]^{\aleph_0})$ for all $\kappa \geq \aleph_2$". By Theorem 3.4, $V^{\mathbb{P}} \models$ "$\mathrm{FRP}(\kappa)$ for all cardinals κ of cofinality $\geq \omega_1$".

(2): Suppose that $V \models$ "ZFC + CH + FRP". In V, let λ be a cardinal such that $\lambda^{\aleph_0} = \lambda$. Then, for $\mathbb{P} = \mathbb{C}_\lambda$, we have $V^{\mathbb{P}} \models 2^{\aleph_0} = \lambda$ and $V^{\mathbb{P}} \models$"FRP". ❑ (Theorem 3.5)

It seems that we can only establish the consistency of FRP + ¬WRP under $2^{\aleph_0} \geq \aleph_3$ by the method as above. However, it is shown in Fuchino, Sakai, Soukup and Usuba [12] using a completely different method that FRP + ¬WRP is also consistent with $2^{\aleph_0} \leq \aleph_2$ modulo some large cardinal.

4 Reflection property of meta-Lindelöfness under FRP

Definition 4.1. We say that a topological space X is *small subspaces meta-Lindelöf* (*ssmL* for short) if every subspace of X of size $\aleph_1$ is meta-Lindelöf.

Remembering the Definition 1.1 of $\aleph_1$-metrizability, the natural wording for this notion might be "$\aleph_1$-meta-Lindelöf". However "$\aleph_1$-meta-Lindelöf" has been already used for a different notion in the literature and hence the present choice of the term "*ssmL*".

Nevertheless, we shall also say for an uncountalble cardinal κ that a topological space X is $<\kappa$-meta-Lindelöf ($\leq\kappa$-meta-Lindelöf resp.) if every subspace Y of X of cardinality $<\kappa$ ($\leq\kappa$ resp.) is meta-Lindelöf.

In the following, $L(X)$ denotes the *Lindelöf number* of the topological space X. That is,

$$L(X) = \min\{\kappa : \text{for any open covering } \mathcal{B} \text{ of } X, \text{ there is a subcovering } \mathcal{C} \subseteq \mathcal{B} \text{ of cardinality } \leq \kappa\}.$$

Theorem 4.1. (1) *Assume that λ is an uncountable cardinal and, for each regular $\omega_1 < \kappa \leq \lambda$, we have* FRP($\kappa$). *Suppose that X is a locally separable countably tight space with $L(X) \leq \lambda$. If X is ssmL, then X is meta-Lindelöf.*

(2) *Assume* FRP. *Suppose that X is a locally separable countably tight space. If X is ssmL, then X is meta-Lindelöf.*

Proof. We shall prove only (1) since it is clear that (2) follows from (1).

If X is locally separable, then every cover of X has a refinement consisting of separable subspaces of X. Thus it is enough to show the following $(*)_\kappa$ for all $\kappa \leq \lambda$.

> $(*)_\kappa$ For any locally separable, countably tight, ssmL space X, if $\mathcal{B}$ is an open cover of X of cardinality κ consisting of separable subspaces of X, then $\mathcal{B}$ has a point countable refinement.

We prove $(*)_\kappa$ by induction on κ. Let $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ be an open cover of X as in $(*)_\kappa$. We want to find a point countable refinement of $\mathcal{B}$.

Case 1. $\kappa = \aleph_0$.

$\mathcal{B}$ itself is a point countable cover of X.

Case 2. κ is regular uncountable.

Let $G_\alpha = \cup\{B_\beta : \beta < \alpha\}$ for $\alpha < \kappa$, and $S = \{\alpha < \kappa : \overline{G_\alpha} \neq G_\alpha\}$.

Claim 4.1.1. *S is non-stationary.*

$\vdash$ We prove first the following weaker assertion:

Subclaim 4.1.1.1. *$S \cap E^\kappa_\omega$ is non-staionary.*

$\vdash$ Suppose, toward a contradiction, that $S \cap E^\kappa_\omega$ were stationary. For each $\alpha \in S \cap E^\kappa_\omega$ let

(4.1) $\quad p_\alpha \in \overline{G_\alpha} \setminus G_\alpha.$

Fix $h : S \cap E^\kappa_\omega \to \kappa$ such that $p_\alpha \in B_{h(\alpha)}$. For each $\alpha < \kappa$ let $D_\alpha \in [B_\alpha]^{\aleph_0}$ be dense in B_α. By $p_\alpha \in \overline{\bigcup_{\beta<\alpha} B_\beta} = \overline{\bigcup_{\beta<\alpha} D_\beta}$ and since X is countably tight, there is $g_0(\alpha) \in [\alpha]^{\aleph_0}$ such that $p_\alpha \in \overline{\cup\{D_\beta : \beta \in g_0(\alpha)\}}$.

Let $g(\alpha) = g_0(\alpha) \cup \{h(\alpha)\}$.

Now apply FRP(κ) to these $S \cap E^\kappa_\omega$ and g to obtain $I \in [\kappa]^{\aleph_1}$ such that

(4.2) $\quad \mathrm{cf}(I) = \omega_1,$

(4.3) $\quad h(\alpha) \in I$ for each $\alpha \in S \cap E^\kappa_\omega \cap I,$

(4.4) $\quad g_0(\alpha) \subseteq I$ for each $\alpha \in S \cap E^\kappa_\omega \cap I,$

(4.5) $\quad$ for any regressive function $f : S \cap E^\kappa_\omega \cap I \to \kappa$ with $f(\alpha) \in g_0(\alpha)$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$, there is $\xi^* \in I$ with $\sup f^{-1}\{\xi^*\} = \sup I$.

Let $Y = \{p_\alpha : \alpha \in S \cap E^\kappa_\omega \cap I\} \cup \bigcup\{D_\beta : \beta \in I\}$.

Since $|\,Y\,| = \aleph_1$, Y is meta-Lindelöf. By (4.3) and (4.4), $\mathcal{G} = \{G_\alpha : \alpha \in I\}$ covers Y. So it follows that $\mathcal{G}$ has a point countable refinement $\mathcal{E}$. For each $\alpha \in S \cap E^\kappa_\omega \cap I$ let $E_\alpha \in \mathcal{E}$ be such that $p_\alpha \in E_\alpha$.

Since E_α is an open neighborhood of p_α and $p_\alpha \in \overline{\cup\{D_\beta : \beta \in g_0(\alpha)\}}$ we have $E_\alpha \cap \cup\{D_\beta : \beta \in g_0(\alpha)\} \neq \emptyset$. Let $f(\alpha) \in g_0(\alpha)$ be such that $E_\alpha \cap D_{f(\alpha)} \neq \emptyset$.

By (4.5), there is $\xi^* \in I$ such that $J = f^{-1}\{\xi^*\}$ is unbounded in I. Since D_{ξ^*} is countable and $E_\alpha \cap D_{\xi^*} \neq \emptyset$ for all $\alpha \in J$, it follows that there is $d \in D_{\xi^*}$ such that $K = \{\alpha \in J : d \in E_\alpha\}$ is unbounded in I.

Since $\mathcal{E}$ is point countable, $\mathcal{E}' = \{E \in \mathcal{E} : d \in E\}$ is countable. Since $E_\alpha \in \mathcal{E}'$ for each $\alpha \in K$, there are $K' \subseteq K$ and $E^* \in \mathcal{E}$ such that K' unbounded in I and $E_\alpha = E^*$ for all $\alpha \in K'$.

Let $\beta \in I$ be such that $E^* \subseteq G_\beta$. Then $E_\gamma \ni p_\gamma \notin G_\beta$ for all $\gamma \in (S \cap E^\kappa_\omega \cap I) \setminus \beta$ by (4.1). In particular, $E_\gamma \neq E^*$ for any $\gamma \in K' \setminus \beta$. This is a contradiction to the choice of E^* and K'. $\dashv$ (Subclaim 4.1.1.1)

Let C be a club subset of κ consisting of limit ordinals such that $S \cap E^\kappa_\omega \cap C = \emptyset$. Let

$$D = \{\alpha \in C : \alpha \setminus S \text{ is cofinal in } \alpha\}.$$

Since D is a club subset of κ, we are done with the following subclaim.

Subclaim 4.1.1.2. *$S \cap D = \emptyset$.*

$\vdash$ Let $\alpha \in D$. If $\mathrm{cf}(\alpha) = \omega$, then $\alpha \notin S$ since $D \subseteq C$. So assume $\mathrm{cf}(\alpha) > \omega$. Suppose that $p \in \overline{G_\alpha}$. By the countable tightness of X, there is an $Y \in [G_\alpha]^{\aleph_0}$ such that $p \in \overline{Y}$. By the definition of D there is a $\beta < \alpha$ such that $G_\beta \supseteq Y$ and $\beta \in \alpha \setminus S$. Then $p \in \overline{G_\beta} = G_\beta \subseteq G_\alpha$. This shows that $\overline{G_\alpha} = G_\alpha$ and hence $\alpha \notin S$. $\dashv$ (Subclaim 4.1.1.2)

$\dashv$ (Claim 4.1.1)

Let C be a club in $\kappa \setminus S$ consisting of limit ordinals and let $\langle \gamma_i : i < \kappa \rangle$ be an increasing enumeration of C. Let $H_i = G_{\gamma_{i+1}} \setminus G_{\gamma_i}$ for $i < \kappa$. Then $\{H_i : i < \omega_1\}$ is a partition of X into clopen sets.

Each H_i is covered by $\mathcal{U}_i = \{B_\xi \setminus G_{\gamma_i} : \gamma_i < \xi < \gamma_{i+1}\}$. Since $|\mathcal{U}_i| < \kappa$, $\mathcal{U}_i$ has a pointwise countable refinement $\mathcal{F}_i$ by the induction hypothesis. Since H_i's are pairwise disjoint, $\mathcal{F} = \cup\{\mathcal{F}_i : i < \kappa\}$ is also point countable. Clearly $\mathcal{F}$ is an open covering of X refining $\mathcal{B}$.

Case 3. κ is singular.

Let $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ be a continuously and strictly increasing cofinal sequence of cardinals in κ. Put $G_i = \bigcup\{B_\alpha : \alpha < \kappa_i\}$ for each $i < \mathrm{cf}(\kappa)$.

By the induction hypothesis, for each $i < \mathrm{cf}(\kappa)$, there is a point countable refinement $\mathcal{C}_i$ of the open covering $\{B_\alpha : \alpha < \kappa_i\}$ of G_i. Note that each element C of $\mathcal{C}_i$ is separable since C is an open subset of some B_α, $\alpha < \kappa_i$.

Put $\mathcal{C} = \bigcup_{i<\mathrm{cf}(\kappa)} \mathcal{C}_i$. Then $\mathcal{C}$ covers X and $\mathrm{ord}(\mathcal{C}) \leq \mathrm{cf}(\kappa)$.

For C, $C' \in \mathcal{C}$, let $C \sim C'$ if and only if $C \cap C' \neq \emptyset$ and let $\approx$ be the transitive closure of $\sim$.

Claim 4.1.2. *Each of the equivalence classes of $\approx$ has cardinality $\leq \mathrm{cf}(\kappa)$.*

$\vdash$ Let $C \in \mathcal{C}$ be arbitrary. Choose a countable dense subset D of C. If $C \sim C'$ then there is $d \in D \cap C'$. Since $\mathrm{ord}(d, \mathcal{C}) \leq \mathrm{cf}(\kappa)$ for $d \in D$ and D is countable, it follows that $|\{C' \in \mathcal{C} : C \sim C'\}| \leq \mathrm{cf}(\kappa)$. Hence $|\{C' \in \mathcal{C} : C \approx C'\}| \leq \mathrm{cf}(\kappa)$ as well. $\dashv$ (Claim 4.1.2)

Let $\mathbb{E}$ be the set of all equivalence classes of the relation $\approx$. Then $\{\cup e : e \in \mathbb{E}\}$ is a partition of X into disjoint open sets, and every $\cup e$ is

covered by e. Since $|\,e\,| \leq \mathrm{cf}(\kappa) < \kappa$, we can apply the induction hypothesis $(*)_\lambda$, $\lambda < \kappa$ to get a point countable refinement $\mathcal{F}_e$ of e which cover $\cup e$. Since e refines $\mathcal{B}$ for each $e \in \mathbb{E}$, $\mathcal{F} = \bigcup\{\mathcal{F}_e : e \in \mathbb{E}\}$ is as desired.
❑ (Theorem 4.1)

The statement of Theorem 1.2 can be slightly modified to obtain the following ZFC result:

Theorem 4.2. *Suppose that X is a locally compact meta-Lindelöf space. If X is $\aleph_1$-metrizable, then X is metrizable.*

Proof. Let $\mathcal{E}$ be a point countable cover of X consisting of open subsets of X with compact closures. By Dow's theorem (Theorem 1.1), $\overline{E}$ is metrizable and hence separable for all $E \in \mathcal{E}$. It follows that every $E \in \mathcal{E}$ is also metrizable and separable.

For E, $E' \in \mathcal{E}$, let $E \sim E'$ if and only if $E \cap E' \neq \emptyset$. Let $\approx$ be the transitive closure of $\sim$.

Similarly to the proof of 4.1.2, we can show that each of the equivalence classes of $\approx$ has cardinality $\leq \aleph_0$ because every $E \in \mathcal{E}$ is separable and the cover $\mathcal{E}$ is point countable.

Let $\mathbb{E}$ be the set of all equivalence classes of the relation $\approx$. Then $\{\cup e : e \in \mathbb{E}\}$ is a partition of X into disjoint open sets.

For $e \in \mathbb{E}$, $\cup e$ is a countable union of open subspaces which are metric spaces. So $\cup e$ is also a metric space by the Bing metrization theorem.

Thus X can be partitioned into clopen subspaces which are all separable metric spaces. It follows that X is also metrizable. ❑ (Theorem 4.2)

The same argument as above also proves:

Proposition 4.3. *Suppose that X is a locally compact meta-Lindelöf space. If X is locally metrizable, then X is metrizable.* ❑

This proposition with the proof similar to the one above seems to be well-known.

Balogh's theorem (Theorem 1.2) can be obtained now as a corollary of Theorems 4.1 and 4.2 under the Fodor-type Reflection Principle:

Corollary 4.4. (1) *Let λ be a cardinal and assume that for each regular $\omega_1 < \kappa \leq \lambda$ we have* $\mathrm{FRP}(\kappa)$. *Suppose that X is a locally countably compact space with $L(X) \leq \lambda$. If X is $\aleph_1$-metrizable, then X is metrizable.*

(2) *Assume* FRP. *Suppose that X is a locally countably compact space. If X is $\aleph_1$-metrizable, then X is metrizable.*

Proof. We prove only (1) since it is clear that (2) follows from (1).

Let X be as in (1). Then every point of X has a countably compact neighborhood, and this neighborhood is metrizable by Dow's theorem (Theorem 1.1). Inparticular, the neighborhood is compact since countalbe compactness and compactness are equivalent for metrizable spaces. Thus the neighberhood is separable because compact metric spaces are separable. It follows also that X is countably tight.

X is ssmL since it is $\aleph_1$-metrizable. Hence X is meta-Lindelöf by Theorem 4.1 (1). By Theorem 4.2, it follows that X is metrizable.

❑ (Corollary 4.4)

As noted in Fact 2.1, FRP($\aleph_1$) is a theorem in ZFC. Thus the proofs of Theorem 4.1 and Corollary 4.4 also establish the following ZFC results:

Corollary 4.5. *Suppose that X is a locally separable countably tight space with $L(X) \leq \aleph_1$. If X is ssmL, then X is meta-Lindelöf.* ❑

Corollary 4.6. *Suppose that X is a locally compact space with $L(X) \leq \aleph_1$. If X is $\aleph_1$-metrizable, then X is metrizable.* ❑

5 Almost metrizability and almost meta-Lindelöfness

The following theorem may be seen as a singular compactness theorem on meta-Lindelöfness of locally separable countably tight spaces analogous to the famous Shelah's Singular Compactness Theorem on the notion of freeness (Shelah [16]). This theorem shows that the regularity of κ in Proposition 1.4 cannot be dropped.

Theorem 5.1. *Suppose that X is a locally separable countably tight space and $|\,X\,|$ is singular. If X is almost meta-Lindelöf then X is meta-Lindelöf.*

The proof of Theorem 5.1 will be given after Lemma 5.4.

Corollary 5.2. *Suppose that X is a locally compact space and $|\,X\,|$ is singular. If X is almost metrizable then X is metrizable.*

Proof. By Theorem 5.1 and Theorem 4.2. (See the argument of the proof of Corollary 4.4.) ❑ (Corollary 5.2)

Proposition 5.3. *Suppose that X is a locally separable almost meta-Lindelöf space. Then every open covering of X of cardinality $< |\,X\,|$ consisting of separable subspaces has a point countable refinement.*

Proof. The proof is quite similar to that of Theorem 4.1. It is enough to prove the following $(*)_\kappa$ for all cardinal κ by induction on κ.

$(*)_\kappa$ For any locally separable countably tight almost meta-Lindelöf space X with $|X| > \kappa$, if $\mathcal{B}$ is an open cover of X of cardinality κ consisting of separable subspaces of X, then $\mathcal{B}$ has a point countable refinement.

Assume that $(*)_{\kappa'}$ holds for all $\kappa' < \kappa$ and let $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ be an open cover of X as in $(*)_\kappa$.

The inductive proof of $(*)_\kappa$ is divided into three cases just as in the proof of Theorem 4.1.

Case 1. $\kappa \le \aleph_0$.

$\mathcal{B}$ itself is a point countable cover of X.

Case 2. κ is regular uncountable.

In this case, we have to prove the following Claim 5.3.1 corresponding to Claim 4.1.1 without FRP.

Let $G_\alpha = \bigcup\{B_\beta : \beta < \alpha\}$ for $\alpha < \kappa$ and $S = \{\alpha < \kappa : \overline{G_\alpha} \ne G_\alpha\}$.

Claim 5.3.1. *S is non-stationary.*

$\vdash$ Toward a contradiction, suppose that S were stationary. For each $\alpha \in S$, let $p_\alpha \in \overline{G_\alpha} \setminus G_\alpha$. For $\alpha \in \kappa$, let D_α be a countable dense subset of B_α. Note that we have $\overline{G_\alpha} = \overline{\bigcup_{\beta<\alpha} D_\beta}$ for $\alpha < \kappa$.

Let $A = \{p_\alpha : \alpha \in S\} \cup \bigcup_{\beta<\kappa} D_\beta$. Since X is almost meta-Lindelöf and $|A| \le \kappa < |X|$, A as a subspace of X is meta-Lindelöf. Thus there is a point countable open refinement $\mathcal{E}$ of the open covering $\{G_\alpha : \alpha < \kappa\}$ of A. For each $\alpha \in S$, choose $E_\alpha \in \mathcal{E}$ such that $p_\alpha \in E_\alpha$. Since E_α is an open neighborhood of p_α and $p_\alpha \in \overline{\bigcup_{\beta<\alpha} D_\beta}$, there is $f(\alpha) < \alpha$ such that $E_\alpha \cap D_{f(\alpha)} \ne \emptyset$.

By the (usual) Fodor lemma, there is a $\beta^* < \kappa$ such that $T = \{\alpha \in S : f(\alpha) = \beta^*\}$ is stationary. Since D_{β^*} is countable, there is a $d^* \in D_{\beta^*}$ such that $\{\alpha \in T : d^* \in D_{\beta^*} \cap E_\alpha\}$ is unbounded in κ. Note that $d^* \in A$ by $D_{\beta^*} \subseteq A$. Hence, by point countability of $\mathcal{E}$ (on A), there is $E^* \in \mathcal{E}$ such that $d^* \in E^*$ and $\{\alpha \in T : E^* = E_\alpha\}$ is unbounded in κ. Let $\gamma < \kappa$ be such that $E^* \subseteq G_\gamma$ and let $\alpha \in \kappa \setminus \gamma$ be such that $E^* = E_\alpha$. Then $p_\alpha \in E_\alpha = E^* \subseteq G_\gamma$. This is a contradiction to the choice of p_α.

$\dashv$ (Claim 5.3.1)

The rest of this case is just as in Case 2 in the proof of Theorem 4.1.

Case 3. κ is singular.

Let $\langle \kappa_i : i < \mathrm{cf}(\kappa)\rangle$ be a continuously and strictly increasing cofinal sequence of cardinals in κ. Put $G_i = \bigcup\{B_\alpha : \alpha < \kappa_i\}$ for $i < \mathrm{cf}(\kappa)$.

For $i < \mathrm{cf}(\kappa)$, if $|\,G_i\,| = |\,X\,|$ there is a point countable refinement $\mathcal{C}_i$ of the open covering $\{B_\alpha : \alpha < \kappa_i\}$ of G_i by the induction hypothesis. If $|\,G_i\,| < |\,X\,|$, there is also a a point countable refinement $\mathcal{C}_i$ of the open covering $\{B_\alpha : \alpha < \kappa\}_i$ of G_i by the almost meta-Lindelöfness of X.

$\bigcup_{i<\mathrm{cf}(\kappa)} C_i$ is then a point countable refinement of $\mathcal{B}$. ❑ (Proposition 5.3)

Lemma 5.4. *Suppose that X is a countably tight separable space. Then every dense set in X has a countable dense subset.*

Proof. Fix a countable dense subset D^* of X. For a given dense subset D of X, we want to find a dense countable $D_0 \subseteq D$. Since X is countably tight, for each $p \in D^*$ there is an $A_p \in [D]^{\aleph_0}$ such that $p \in \overline{A_p}$. Then $\bigcup_{p\in D^*} A_p$ is a countable dense subset of D. ❑ (Lemma 5.4)

Proof of Theorem 5.1: We may assume that the underlying set of X is a singular cardinal λ.

let us first show that, for every open covering $\mathcal{B}$ of X, there is an open covering $\mathcal{B}'$ refining $\mathcal{B}$ with $\mathrm{ord}(\mathcal{B}') \leq \mathrm{cf}(\lambda)$.

For each $\alpha \in X$, let O_α be an open neighborhood of α which is a separable subspace of X. Let D_α be a countable dense subset of O_α with

(5.1) $\alpha \in D_\alpha$.

Fix an increasing sequence of regular cardinals $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ cofinal in λ. For each $i < \mathrm{cf}(\lambda)$, let $P_i = \bigcup_{\alpha<\lambda_i} O_\alpha$ and $E_i = \bigcup_{\alpha<\lambda_i} D_\alpha$. Then P_i is an open subset of X and E_i is a dense subset of P_i of size $\leq \lambda_i$. By almost meta-Lindelöfness of X, each E_i is meta-Lindelöf and thus there is a refinement $\mathcal{B}_i$ of $\{O \cap P_i : O \in \mathcal{B}\} \setminus \{\emptyset\}$ such that

(5.2) $E_i \subseteq \bigcup \mathcal{B}_i$

(5.3) $\mathrm{ord}(\alpha, \mathcal{B}_i) \leq \aleph_0$ for all $\alpha \in E_i$.

Note that we do not require $P_i \subseteq \bigcup \mathcal{B}_i$. However, since $\lambda_i \subseteq E_i$ by (5.1), it follows from (5.2) that $\mathcal{B}' = \bigcup_{i<\mathrm{cf}(\lambda)} \mathcal{B}_i$ is an open covering of X refining $\mathcal{B}$. Thus, the next claim implies that $\mathcal{B}'$ is as desired.

Claim 5.4.1. $\mathrm{ord}(\alpha, \mathcal{B}_i) \leq \aleph_0$ *for every $i < \mathrm{cf}(\lambda)$ and $\alpha \in P_i$.*

$\vdash$ Suppose that $\alpha \in P_i$. Since O_α is separable, so is $P_i \cap O_\alpha$; since X is first countable, so is $P_i \cap O_\alpha$.

Since E_i is dense in P_i, $E_i \cap P_i \cap O_\alpha$ is dense in $P_i \cap O_\alpha$. By Lemma 5.4, there is a countable dense set $E \subseteq E_i \cap P_i \cap O_\alpha$. Now suppose $\mathrm{ord}(\alpha, \mathcal{B}') > \aleph_0$. Since E is countable dense set in open neighborhood $P_i \cap O_\alpha$ of α, there is $\beta \in E$ such that $\mathrm{ord}(\beta, \mathcal{B}') > \aleph_0$. Since $E \subseteq E_i$, this is a contradiction to (5.3). $\dashv$ (Claim 5.4.1)

Now, we are ready to show that X is meta-Lindelöf. Let $\mathcal{B}$ be an open covering. We may assume that every element of $\mathcal{B}$ is a first countable separable subspace. By what we proved above, there is an open covering $\mathcal{B}'$ refining $\mathcal{B}$ with $\mathrm{ord}(\mathcal{B}') \leq \mathrm{cf}(\lambda)$. Let $\sim$ be the binary relation on $\mathcal{B}'$ defined by $O \sim O'$ if and only if $O \cap O' \neq \emptyset$ and let $\approx$ be the transitive closure of $\sim$. Let $\mathbb{E}$ be the set of equivalence classes of $\approx$. Then, by the same argument as that of Claim 4.1.2, we can show that $|\,e\,| \leq \mathrm{cf}(\lambda)$ for all $e \in \mathbb{E}$. Since $\bigcup e$, $e \in \mathbb{E}$ are pairwise disjoint open sets, it is enough to show that there is a point countable refinement of the open covering e of $\bigcup e$ for all $e \in \mathbb{E}$. If $|\bigcup e\,| < \lambda$, then we are done since $\bigcup e$ meta-Lindelöf is meta-Lindelöf by almost meta-Lindelöfness of X. If $|\bigcup e\,| = \lambda$, we are done by applying Proposition 5.3. ❑ (Theorem 5.1)

Proposition 5.3 also has the following obvious application:

Corollary 5.5. *Suppose that X is a locally separable space with $L(X) < |\,X\,|$. If X is almost meta-Lindelöf then X is meta-Lindelöf.*

Proof. By Proposition 5.3. ❑ (Corollary 5.5)

Corollary 5.6. *Suppose that X is locally compact space with $L(X) < |\,X\,|$. If X is almost metrizable then X is metrizable.*

Proof. By Corollary 5.5 and the argument of the proof of Corollary 4.4. ❑ (Corollary 5.6)

In contrast to the results we have seen above, the anticompactness of metrizability holds in ZFC for any uncountable cardinal, if we consider all topological spaces which are not necessarily locally compact:

Proposition 5.7. *For any uncountable cardinal κ, there is a topological space $X = (X, \tau)$ of cardinality κ which is paracompact and almost metrizable but not metrizable.*

Proof. The construction of a space X with the property as above is divided into two cases:

Case 1. κ is regular.

Let $X = \kappa + 1$ and let τ be the topology generated from the basis $\{\{\alpha\} : \alpha < \kappa\} \cup \{[\alpha, \kappa] : \alpha < \kappa\}$ where $[\alpha, \kappa]$ denotes the closed interval $\{\beta : \alpha \leq \beta \leq \kappa\}$.

Since κ is regular, $\chi(\lambda, X) = \kappa$. In particular, X is not metrizable.

If $Y \in [X]^{<\kappa}$, Y is discrete: If $\kappa \notin Y$ then this is clear. If $\kappa \in Y$ it is enough to show that κ is an isolated point in Y. Let $\alpha^* = (\sup Y \setminus \{\kappa\}) + 1$. Since κ is regular we have $\alpha^* < \kappa$. Then $[\alpha^*, \kappa] \cap Y = \{\kappa\}$.

In particular, every $Y \in [X]^{<\kappa}$ is metrizable and thus X is almost metrizable.

If $\mathcal{E}$ is an open covering of X, there is an $O \in \mathcal{E}$ and $\alpha^* < \kappa$ such that $[\alpha^*, \kappa] \subseteq O$. $\mathcal{E}' = \{\{\alpha\} : \alpha < \alpha^*\} \cup \{[\alpha^*, \kappa]\}$ is then a pairwise disjoint open covering of X refining $\mathcal{E}$. This shows that X is paracompact.

Case 2. κ is singular.

Let $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ be a strictly increasing cofinal sequence of regular cardinals in κ. For each $f \in \prod_{i<\mathrm{cf}(\kappa)} \kappa_i$ and $j < \mathrm{cf}(\kappa)$, let

$$O_{f,j} = \bigcup\{(f(k), \kappa_k) : j < k < \mathrm{cf}(\kappa)\} \cup \{\kappa\}$$

where $(f(k), \kappa_k)$ denotes the open interval $\{\alpha : f(k) < \alpha < \kappa_k\}$. Note that

(5.4) for any f, $g \in \prod_{i<\mathrm{cf}(\kappa)} \kappa_i$ and $j < \mathrm{cf}(\kappa)$, $O_{f,j} \subseteq O_{g,k}$ for some $k < \mathrm{cf}(\kappa)$ if and only if $f \leq^* g$, that is, if f is (modulo sets of size $< \mathrm{cf}(\kappa)$) almost dominated by g.

Let $X = \kappa + 1$ and let τ be the topology generated from the basis $\{\{\alpha\} : \alpha < \kappa\} \cup \{O_{f,j} : f \in \prod_{i<\mathrm{cf}(\kappa)} \kappa_i, j < \mathrm{cf}(\kappa)\}$.

We claim that $X = (X, \tau)$ is as desired.

First, we show that X is almost metrizable. As in Case 1, this follows from the fact that every $Y \subseteq [X]^{<\kappa}$ is a discrete space: It is enough to consider the case that $\kappa \in Y$ and to show that κ is a isolated point in Y. Let $i^* < \mathrm{cf}(\kappa)$ be such that $|\,Y\,| < \kappa_{i^*}$. Then $\kappa_i \cap Y$ is bounded in κ_i for all $i^* \leq i < \mathrm{cf}(\kappa)$. Let $f^* \in \prod_{i<\mathrm{cf}(\kappa)} \kappa_i$ be defined by

$$f^*(i) = \begin{cases} (\sup \kappa_i \cap Y) + 1 & \text{if } i^* \leq i < \mathrm{cf}(\kappa), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $O_{f^*,i^*} \cap Y = \{\kappa\}$.

To show that X is not metrizable, it is enough to show that $\chi(\kappa, X) \geq \kappa^+$.

Suppose that $\mathcal{E}$ is a family of open neighborhoods of κ with $|\,\mathcal{E}\,| \leq \kappa$. We may assume without loss of generality that every element of $\mathcal{E}$ is of the form $O_{f,j}$ for some $f \in \prod_{i<\mathrm{cf}(\kappa)} \kappa_i$ and $j < \mathrm{cf}(\kappa)$. Let

$$\mathcal{F} = \{f \in \textstyle\prod_{i<\mathrm{cf}(\kappa)} \kappa_i \,:\, O_{f,j} \in \mathcal{E} \text{ and } f(\ell) = 0 \text{ for all } \ell < j \\ \text{for some } j < \mathrm{cf}(\kappa)\}.$$

Since $|\,\mathcal{F}\,| \leq \kappa$, $\mathcal{F}$ is not cofinal in $\prod_{i<\mathrm{cf}(\kappa)} \kappa_i$ with respect to $\leq^*$. Let $g^* \in \prod_{i<\mathrm{cf}(\kappa)} \kappa_i$ be such that $g^* \not\leq^* f$ for all $f \in \mathcal{F}$. By (5.4), it follows that $O_{f,i} \not\subseteq O_{g^*,0}$ for all $O_{f,i} \in \mathcal{E}$. Thus $\mathcal{E}$ is not a neighborhood basis of κ.

Paracompactness of X can be shown similarly to Case 1.

❑ (Proposition 5.7)

Proposition 5.7 shows that the local compactness of the space X in Theorem 4.1, Corollary 5.2 or Corollary 5.6 cannot be completely dismissed.

Extending the terminology of local countablility, let us say that a topological space X is locally $< \kappa$ if each $x \in X$ has a neighborhood U of cardinality $< \kappa$. Locally $\leq \kappa$ and locally κ spaces are defined similarly.

Corollary 5.10 below is a variant of Theorem 4.1 where the countable compactness of the space X is replaced by "locally $\leq \aleph_1$". Since any compact metric space has cardinality $\leq 2^{\aleph_0}$, this corollary generalizes Theorem 4.1 under CH.

For a covering $\mathcal{B}$ of a space X and a cardinal μ, we say $\mathrm{ord}(\mathcal{B}) \leq \mu$ *and μ is not attained* if $\mathrm{ord}(x, \mathcal{B}) < \mu$ for all $x \in X$. Note that for a successor μ this is just equivalent to $\mathrm{ord}(\mathcal{B}) < \mu$.

Proposition 5.8. *Assume that μ is a regular cardinal $> \aleph_1$, $\mu \leq \lambda$ and* $\mathrm{FRP}(\delta)$ *holds for all regular $\mu \leq \delta \leq \lambda$. If X is a countably tight locally $< \mu$, $< \mu$-meta-Lindelöf space of size $\leq \lambda$ and $\mathcal{B}$ is an open covering of X consisting of open sets of size $< \mu$ then there is a refinement $\mathcal{B}'$ of $\mathcal{B}$ such that* $\mathrm{ord}(\mathcal{B}') \leq \mu$ *and μ is not attained.*

Proof. For cardinal $\kappa \leq \lambda$, let $(*)_\kappa$ be the following assertion:

- $(*)_\kappa$ For any countably tight, $<\mu$-meta-Lindelöf space X and any open covering $\mathcal{B}$ of X of size κ consisting of open sets of size $< \mu$, there is a refinement $\mathcal{B}'$ of $\mathcal{B}$ such that $\mathrm{ord}(\mathcal{B}') \leq \mu$ and μ is not attained.

Since X considered in this proposition is is locally $< \mu$, it is enough to show that $(*)_\kappa$ holds for all $\kappa \leq \lambda$. We shall prove this by induction on κ.

Case 1. $\kappa < \mu$.

The assertion $(*)_\kappa$ in this case is trivial.

Case 2. κ is a regular cardinal $\geq \mu$.

Assume that $(*)_{\kappa'}$ holds for all $\kappa' < \kappa$. Let X and $\mathcal{B}$ be as in $(*)_\kappa$. Note that we have $|\,X\,| \leq |\,\mathcal{B}\,| \cdot \mu = \kappa$.

Case 2a. $|\,X\,| = \kappa$.

Without loss of generality we may assume that $X = \kappa$. Let $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$. For $\alpha < \kappa$, let $G_\alpha = \bigcup_{\beta<\alpha} B_\beta$. Since $|\,B_\alpha\,| < \mu$ for all $\alpha < \kappa$, we have $|\,G_\alpha\,| < \kappa$ for all $\alpha < \kappa$. Thus $C = \{\alpha < \kappa : \alpha \text{ is a limit and } G_\alpha = \alpha\}$ is a club.

Claim 5.8.1. $\{\alpha \in C : \overline{G_\alpha} = G_\alpha\}$ *contains a club.*

$\vdash$ Since X is countably tight, we may apply the argument in the proof of Subclaim 4.1.1.2 to see that it is enough to show that $S = \{\alpha \in C \cap E^\kappa_\omega : \overline{G_\alpha} \neq G_\alpha\}$ is non-stationary.

Suppose, toward a contradiction, that S were stationary. For $\alpha \in S$, let

$$p_\alpha \in \overline{G_\alpha} \setminus G_\alpha. \tag{5.5}$$

Since $G_\alpha = \alpha$ and X is countably tight, there is $g_0(\alpha) \in [\alpha]^{\aleph_0}$ such that $p_\alpha \in \overline{g_0(\alpha)}$. Let

$$g(\alpha) = g_0(\alpha) \cup \{p_\alpha\}. \tag{5.6}$$

By FRP(κ), there is $I \in [\kappa]^{\aleph_1}$ as in the definition of FRP(κ) for these S and g. For every $\alpha \in S \cap I$, we have $p_\alpha \in I$ and $g_0(\alpha) \subseteq I$ by (2.2) and (5.6). Since $|\,I\,| = \aleph_1 < \mu$, I as a subspace of X is meta-Lindelöf. Since $S \cap I$ is cofinal in I, $\mathcal{E} = \{G_\alpha \cap I : \alpha \in I\}$ is an open covering of I. It follows that there is a point countable refinement $\mathcal{E}^*$ of $\mathcal{E}$. For each $\alpha \in S \cap I$, let $E_\alpha \in \mathcal{E}^*$ be such that $p_\alpha \in E_\alpha$. Since $E_\alpha \cap I$ is an open neighborhood of p_α in I, $g_0(\alpha) \subseteq I$ and $p_\alpha \in \overline{g(\alpha)}$, we have $g_0(\alpha) \cap E_\alpha \neq \emptyset$. Let $f(\alpha) \in g_0(\alpha) \cap E_\alpha$. Then $g : S \cap I \to I$ is a regressive function and hence there is $\beta^* \in I$ such that $\{\alpha \in S \cap I : f(\alpha) = \beta^*\}$ is stationary in $\sup(I)$.

Since $\mathrm{ord}(\beta^*, \mathcal{E}^*) \leq \aleph_0$, there is $E^* \in \mathcal{E}^*$ such that $J = \{\alpha \in S \cap I : E_\alpha = E^*\}$ is unbounded in $\sup(I)$. Let $\gamma \in I$ be such that $E^* \in G_\gamma$ and let $\alpha \in J \setminus \gamma$. Then $p_\alpha \in E_\alpha = E^* \subseteq G_\gamma \subseteq G_\alpha$. This is a contradiction to (5.5). $\dashv$ (Claim 5.8.1)

The rest of the proof is almost the same as in Case 2 of the proof of Theorem 4.1.

Let D be a club subset of $\{\alpha \in C : \overline{G_\alpha} = G_\alpha\}$ and let $\langle \gamma_i : i < \kappa \rangle$ be an increasing enumeration of D. Let $H_i = G_{\gamma_{i+1}} \setminus G_{\gamma_i}$ for all $i < \kappa$. Then $\{H_i : i < \kappa\}$ is a partition of X into disjoint open sets.

For $i < \kappa$, $\mathcal{B}_i = \{B_\alpha \cap H_i : \gamma_i \leq \alpha < \gamma_{i+1}\}$ is an open covering of H_i and $|\mathcal{B}_i| < \kappa$. Thus, by the induction hypothesis, there is a refinement $\mathcal{B}'_i$ of $\mathcal{B}_i$ such that $\mathrm{ord}(\mathcal{B}'_i) \leq \mu$ and μ is not attained. $\mathcal{B}' = \bigcup_{i<\kappa} \mathcal{B}'_i$ is then a refinement of $\mathcal{B}$ such that $\mathrm{ord}(\mathcal{B}') \leq \mu$ and μ is not attained.

Case 2b. $|X| < \kappa$.

For each $p \in X$, choose a $B_p \in \mathcal{B}$ such that $p \in B_p$. Then $\mathcal{B}' = \{B_p : p \in X\}$ is a refinement of $\mathcal{B}$ and $|\mathcal{B}'| < \kappa$. Hence, by induction hypothesis, there is a refinement $\mathcal{B}''$ of $\mathcal{B}'$ such that $\mathrm{ord}(\mathcal{B}'') \leq \mu$ and μ is not attained.

Case 3. κ is a singular cardinal $\geq \mu$.

The proof of this case is quite similar to Case 3 in the proof of Theorem 4.1.

Let $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ be a continuously and strictly increasing sequence of cardinals cofinal in κ. Put $G_i = \bigcup\{B_\alpha : \alpha < \kappa_i\}$ for each $i < \mathrm{cf}(\kappa)$.

By the induction hypothesis, for each $i < \mathrm{cf}(\kappa)$, there is a refinement $\mathcal{C}_i$ of the open covering $\{B_\alpha : \alpha < \kappa_i\}$ of G_i such that $\mathrm{ord}(\mathcal{C}_i) \leq \mu$ and μ is not attained. Note that each element C of $\mathcal{C}_i$ is of cardinality $< \mu$.

Put $\mathcal{C} = \bigcup_{i<\mathrm{cf}(\kappa)} \mathcal{C}_i$. Then $\mathcal{C}$ covers X and $\mathrm{ord}(\mathcal{C}) \leq \max\{\mathrm{cf}(\kappa), \mu\}$.

For C, $C' \in \mathcal{C}$, let $C \sim C'$ if and only if $C \cap C' \neq \emptyset$ and let $\approx$ be the transitive closure of $\sim$.

Claim 5.8.2. *Each of the equivalence classes with respect to $\approx$ has cardinality $\leq \max\{\mathrm{cf}(\kappa), \mu\}$.*

$\vdash$ Let $C \in \mathcal{C}$ be arbitrary. If $C \sim C'$ then there is $d \in C \cap C'$. Since $\mathrm{ord}(d, \mathcal{C}) \leq \max\{\mathrm{cf}(\kappa), \mu\}$ for $d \in C$ and C is of cardinality $< \mu$, it follows that $|\{C' \in \mathcal{C} : C \sim C'\}| \leq \mathrm{cf}(\kappa) \cdot \mu = \max\{\mathrm{cf}(\kappa), \mu\}$. Hence $|\{C' \in \mathcal{C} : C \approx C'\}| \leq \max\{\mathrm{cf}(\kappa), \mu\}$ as well. $\dashv$ (Claim 5.8.2)

Let $\mathbb{E}$ be the set of all equivalence classes of the relation $\approx$. Then $\{\cup e : e \in \mathbb{E}\}$ is a partition of X into disjoint open sets, and every $\cup e$ is covered by e. Since $|e| \leq \max\{\mathrm{cf}(\kappa), \mu\} < \kappa$ we can apply the induction hypothesis $(*)_\lambda$, $\lambda < \kappa$ to get a refinement $\mathcal{F}_e$ of e which cover $\cup e$ such that $\text{ord}(\mathcal{F}_e) \leq \mu$ and μ is not attained. Since e refines $\mathcal{B}$ for each $e \in \mathbb{E}$, $\mathcal{F} = \bigcup\{\mathcal{F}_e : e \in \mathbb{E}\}$ is as desired. ❑ (Proposition 5.8)

The following theorem shows that Question 4.3 in [6, Question 4.3] can be (consistently) irrelevant:

Theorem 5.9. (1) *Assume that μ is a regular uncountable cardinal, $\mu \leq \lambda$ and* FRP(δ) *holds for all regular $\mu \leq \delta \leq \lambda$. Suppose that X is a locally $< \mu$ space of cardinality $\leq \lambda$. If X is $< \mu$-metrizable, then X is metrizable.*

(2) *Suppose that μ is a regular uncountable cardinal. Assume* FRP. *If X is locally $< \mu$ and $< \mu$-metrizable then X is metrizable.*

Proof. Again we prove only (1).

Let X be locally $< \mu$ and $< \mu$-metrizable. By Proposition 5.8 and by an argument similar to the one in the proof of Theorem 4.2, X can be partitioned into open subsets of cardinality $< \mu$. By $< \mu$-metrizability of X each open set in the partition is metrizable. Thus it follows that X is metrizable. ❑ (Theorem 5.9)

A. Dow proved the following under Axiom R (see [6, Propositon 6.1]):

Corollary 5.10. (1) *Assume that* FRP(δ) *holds for all regular $\aleph_1 \leq \delta \leq \lambda$. Suppose that X is a locally $\leq \aleph_1$ space of cardinality $\leq \lambda$. If X is $\leq \aleph_1$ metrizable, then X is metrizable.*

(2) *Assume* FRP. *Suppose that X is a locally $\leq \aleph_1$ space. If X is $\aleph_1$-metrizable, then X is metrizable.* ❑

The following natural problem remains open. Note that Dow [6, Theorem 4.2] gives a partial answer to the problem.

Problem 1. *Can Theorem 5.9 also hold for singular μ under some set-theoretic assumptions?*

Though it would sound like a pun, the compactness problem concerning metrizability is also connected to compact cardinals:

Theorem 5.11. (1) *Assume that $\kappa \leq \lambda$ and κ is λ-compact. Suppose that X is of cardinality $\leq \lambda$ and $\chi(p, X) < \kappa$ for all $p \in X$. If X is $< \kappa$-metrizable, then X is metrizable.*

(2) *Assume that κ is compact. Suppose that X is a topological space with $\chi(p, X) < \kappa$ for all $p \in X$. If X is $< \kappa$-metrizable, then X is metrizable.*

(3) *Assume that κ is measurable. Suppose that X is a topological space of cardinality κ with $\chi(p, X) < \kappa$ for all $p \in X$. If X is almost metrizable, then X is metrizable.*

Proof. (1): Let $X = (X, \tau)$. Without loss of generality we may assume that X is of cardinality λ and (the underlying set of) $X = \lambda$.

Let U be a κ-complete fine ultrafilter on $\mathcal{P}_\kappa\lambda$. For each $x \in \mathcal{P}_\kappa\lambda$, let d_x be a metric of the subspace x of X. Let $d : \lambda \times \lambda \to \mathbb{R}^+$ be defined by

(5.7) $\quad d(\alpha, \beta) = r$ if and only if $\{x \in \mathcal{P}_\kappa\lambda : d_x(\alpha, \beta) = r\} \in U$.

It is easy to check that d is a metric on X. Let τ_d be the topology on X incuced by d. We show that $\tau = \tau_d$.

For $\alpha \in \lambda$ and $r \in \mathbb{R}^+$, let $O_d(\alpha, r)$ be the open ball arround α with radius r with respect to the metric d. That is,

$$O_d(\alpha, r) = \{\beta \in \lambda : d(\alpha, \beta) < r\}.$$

Suppose first that $O \in \tau$ and $\alpha \in O$. Let $A = \{x \in \mathcal{P}_\kappa\lambda : \alpha \in x\}$. Since U is fine, we have $A \in U$. Since $O \cap x$ is open in $x \in A$, there is $r_x \in \mathbb{R}^+$ such that $O_x = \{\beta \in x : d_x(\beta, \alpha) < r_x\} \subseteq O \cap x \subseteq O$. By the κ-completeness of U, there is r^* such that $B = \{x \in A : r_x = r^*\} \in U$.

Claim 5.11.1. *$O_d(\alpha, r^*) \subseteq O$.*

$\vdash$ Suppose that $\beta \in O_d(\alpha, r^*)$. Let $r_0 = d(\alpha, \beta)$. Then $r_0 < r^*$. By (5.7) and since $B \in U$, $B' = \{x \in B : \beta \in x$ and $d_x(\alpha, \beta) = r_0\} \in U$. Let $x_0 \in B'$. Since $r_{x_0} = r^* > r_0$, we have $\beta \in O_{x_0} \subseteq O$. $\dashv$ (Claim 5.11.1)

Since $\alpha \in O$ was arbitrary, it follows that $O \in \tau_d$.

Suppose now that $O \in \tau_d$ and $\alpha \in O$. Then there is $r^\dagger \in \mathbb{R}^+$ such that

(5.8) $\quad O_d(\alpha, r^\dagger) \subseteq O$.

By the assumption on (X, τ), there is a neighborhood basis $\{O_n : n < \mu\}$ of α in (X, τ) for some $\mu < \kappa$.

Let $A = \{x \in \mathcal{P}_\kappa\lambda : \alpha \in x\} \in U$. For each $x \in A$, $O_{d_x}(\alpha, r^\dagger) = \{\beta \in x : d_x(\alpha, \beta) < r^\dagger\}$ is an open neighborhood of α in x. Hence there is $n_x < \mu$ such that

(5.9) $\quad O_{n_x} \cap x \subseteq O_{d_x}(\alpha, r^\dagger)$.

By the κ-compactness of U there is an $n^\dagger < \mu$ such that $B = \{x \in A : n_x = n^\dagger\} \in U$.

Claim 5.11.2. *$\alpha \in O_{n^\dagger} \subseteq O$.*

$\vdash$ $\alpha \in O_{n^\dagger}$ is clear by the choice of O_n's.

Suppose that $\beta \in O_{n^\dagger}$. Then $B' = \{x \in B : \beta \in x\} \in U$.

Let $r_0 = d(\alpha, \beta)$. By (5.7) there is an $x_0 \in B'$ such that $d_{x_0}(\alpha, \beta) = r_0$. Since $\beta \in O_{n^\dagger} \cap x_0 = O_{n_{x_0}} \cap x_0 \subseteq O_{d_{x_0}}(\alpha, r^\dagger)$ by (5.9), we have $r_0 < r^\dagger$. By (5.8), it follows that $\beta \in O$. $\dashv$ (Claim 5.11.2)

Since $\alpha \in O$ was arbitrary, it follows that $O \in \tau$.

(2) and (3) follow immediately from (1). ❑ (Theorem 5.11)

Problem 2. *Is the following assertion* (5.10) *consistent?*

(5.10) *Every first countable $\aleph_1$-metrizable space is metrizable.*

Problem 3. *Does* Corollary 5.2 *hold with local compactness replaced by first countability?*

6 Anticompactness of metrizability for locally compact spaces without non-reflecting stationary sets

For a regular cardinal κ, let $\mathrm{ADS}^-(\kappa)$ be the following principle:

$\mathrm{ADS}^-(\kappa)$: there are stationary $S \subseteq \kappa$ and a sequence $\langle a_\alpha : \alpha \in S \rangle$ such that

(6.1) $a_\alpha \subseteq \alpha$ and $\mathrm{otp}(a_\alpha) = \omega$ for all $\alpha \in S$;

(6.2) for any $\beta < \kappa$, there is a mapping $f : S \cap \beta \to \beta$ such that $f(\alpha) < \sup(a_\alpha)$ for all $\alpha \in S \cap \beta$ and $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ are pairwise disjoint.

Let $\mathrm{ADS}^{-*}(\kappa)$ be the assertion that there are a stationary $S \subseteq E^\kappa_\omega$ and a sequence $\langle a_\alpha : \alpha \in S \rangle$ such that (6.1) and (6.2) hold.

Lemma 6.1. *For any regular κ $\mathrm{ADS}^-(\kappa)$ is equivalent to $\mathrm{ADS}^{-*}(\kappa)$.*

Proof. It is clear that $\mathrm{ADS}^-(\kappa)$ follows from $\mathrm{ADS}^{-*}(\kappa)$. So assume $\mathrm{ADS}^-(\kappa)$ and let $S \subseteq \kappa$ and $\langle a_\alpha : \alpha \in S \rangle$ be as in the definition of $\mathrm{ADS}^-(\kappa)$. We have to show that there are a stationary $S^* \subseteq E^\kappa_\omega$ and a sequence $\langle a^*_\alpha : \alpha \in S^* \rangle$ such that they satisfy (6.1) and (6.2).

Case 1. $\{\sup(a_\alpha) : \alpha \in S'\}$ is bounded in κ for some stationary $S' \subseteq S$.

Let $\mathcal{F} = \{a_\alpha : \alpha \in S'\}$ and $\alpha^* = \sup\{\sup(a) : a \in \mathcal{F}\}$. We have $\alpha^* < \kappa$ by assumption. Let $S^* = E^\kappa_\omega \setminus \alpha^*$ and let $\langle a^*_\alpha : \alpha \in S^*\rangle$ be any enumeration of $\mathcal{F}$. Then this S^* and $\langle a^*_\alpha : \alpha \in S^*\rangle$ are as desired.

Case 2. $\{\sup(a_\alpha) : \alpha \in S'\}$ is unbounded in κ for any stationary $S' \subseteq S$.

Claim 6.1.1. $S^* = \{\alpha \in S : \sup(a_\alpha) = \alpha\}$ *is stationary.*

$\vdash$ Otherwise there is a club C such that $C \cap S^* = \emptyset$. For $\alpha \in S \cap C$, $\sup(a_\alpha) < \alpha$. By Fodor's Lemma, there is a stationary $S' \subseteq S \cap C$ and $\delta < \kappa$ such that $\sup(a_\alpha) = \delta$ for all $\alpha \in S'$ which is a contradiction to the assumption of this case. $\dashv$ (Claim 6.1.1)

Clearly $S^* \subseteq E^\kappa_\omega$. Thus these S^* and $\langle a_\alpha : \alpha \in S^*\rangle$ are as desired. ❑ (Lemma 6.1)

Proposition 6.2. *For a regular cardinal* $\kappa > \aleph_1$, $\mathrm{ADS}^-(\kappa)$ *implies* $\neg\mathrm{FRP}(\kappa)$.

Proof. Suppose that a stationary $S \subseteq \kappa$ and a sequence $\langle a_\alpha : \alpha \in S\rangle$ witness $\mathrm{ADS}^-(\kappa)$. By Lemma 6.1, we may assume that $S \subseteq E^\kappa_\omega$. Let $g : S \to [\kappa]^{\aleph_0}$, $\alpha \mapsto a_\alpha$. Then, for any $I \in [\kappa]^{\aleph_1}$, since $\sup(I) < \kappa$, there is an $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$ and $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap I$ are pairwise disjoint. In particular, f is regressive but one to one. This shows that $\mathrm{FRP}(\kappa)$ does not hold. ❑ (Proposition 6.2)

$\mathrm{ADS}^-(\kappa)$ for a regular $\kappa > \aleph_1$ not only negates $\mathrm{FRP}(\kappa)$ but actually it also implies the existence of a topological space as in Proposition 1.3.

Proposition 6.3. *Suppose that* $\mathrm{ADS}^-(\kappa)$ *holds for a regular uncountable* κ*. Then there is a locally countable locally compact space* X *of cardinality* κ *which is almost metrizable but not meta-Lindelöf.*

Proof. Let $\langle a_\alpha : \alpha \in S\rangle$ be a sequence as in the definition of $\mathrm{ADS}^-(\kappa)$. Without loss of generality we may assume that $S \subseteq \mathit{Lim}(\kappa)$ and $a_\alpha \subseteq \{\xi + 1 : \xi \in \kappa\}$. For the latter condition note that we may replace the sequence $\langle a_\alpha : \alpha \in S\rangle$ by $\langle a'_\alpha : \alpha \in S\rangle$ where $a'_\alpha = \{\xi + 1 : \xi \in a_\alpha\}$ for all $\alpha \in S$.

Let $X = S \cup \{\xi + 1 : \xi \in \kappa\}$ with the topology defined by (1.5) and (1.6) in the proof of Proposition 1.3. Then, just as in the proof of Proposition 1.3, we can show that X is not meta-Lindelöf, locally countable and locally compact. Thus the following claim shows that the topological space X is as desired.

Claim 6.3.1. *X is almost metrizable.*

$\vdash$ It is enough to show that $X \cap \beta$ for every $\beta < \kappa$ is metrizable. For $\beta < \kappa$, there is an $f : S \cap \beta \to \beta$. such that $f(\alpha) < \sup(a_\alpha)$ for all $\alpha \in S \cap \beta$ and $a_\alpha \cap f(\alpha)$, $\alpha \in S \cap \beta$ are pairwise disjoint. Let $I = \{\xi + 1 : \xi \in \kappa\} \setminus \bigcup\{a_\alpha \setminus f(\alpha) : \alpha \in S \cap \beta\}$. Then $\mathcal{U} = \{a_\alpha \setminus f(\alpha) : \alpha \in S \cap \beta\} \cup \{\{\alpha\} : \alpha \in I\}$ is an open partition of $X \cap \beta$. Since each element of $\mathcal{U}$ is clearly metrizable, it follows that $X \cap \beta$ is metrizable as well. $\dashv$ (Claim 6.3.1)

❑ (Proposition 6.3)

The following principle ADS(λ) was studied by S. Shelah in [15].

ADS(λ): there is a sequence $\langle a_\alpha : \alpha < \lambda^+ \rangle$ such that

(6.3) $a_\alpha \subseteq \lambda$, $\sup(a_\alpha) = \lambda$ and $\mathrm{otp}(a_\alpha) = \mathrm{cf}(\lambda)$ for all $\alpha < \lambda^+$;

(6.4) for any $\beta < \lambda^+$, there is a mapping $f : \beta \to \lambda$ such that $a_\alpha \setminus f(\alpha)$, $\alpha < \beta$ are pairwise disjoint.

The following is clear from the definition of the principles ADS(λ) and ADS$^-(\lambda^+)$:

Proposition 6.4. *Suppose that* $\mathrm{cf}(\lambda) = \omega$. *Then* ADS($\lambda$) *implies* ADS$^-(\lambda^+)$.

❑

Let us denote with ORP(κ) the assertion that the stationarity of every stationary subset of E^κ_ω reflects down to an ordinal of cofinality ω_1 (in the notation of [5], this is $\mathrm{Refl}(1, E^\kappa_\omega, \omega_1)$). Clearly FRP($\kappa$) implies ORP($\kappa$).

Theorem 6.5. (J. Cummings, M. Foreman and M. Magidor [5, Theorem 7 and Theorem 21])
(1) $\square^*_\lambda$ *implies* ADS(λ).

(2) *If* ZFC + *"there exist ω many supercompact cardinals" is consistent, then it is consistent with* ZFC *that* $\square^*_{\omega_\omega}$ *holds while* ORP($\aleph_{\omega+1}$) *also holds.*

[5] proves actually the consistency of $\square^*_{\omega_\omega}$ together with a reflection property much stronger than ORP($\aleph_{\omega+1}$).

Corollary 6.6. *It is consistent (modulo the large cardinal assumption like the one in Theorem 6.5,(2)) that there is a locally countable locally compact space X of cardinality $\aleph_{\omega+1}$ which is almost metrizable but not meta-Lindelöf (hence in particular* FRP($\aleph_{\omega+1}$) *does not hold), while* ORP($\aleph_{\omega+1}$) *holds.*

Proof. By Theorem 6.5, (1),(2) and Proposition 6.3. ❑ (Corollary 6.6)

In Fuchino, Sakai, Soukup and Usuba [12], the consistency of ORP($\aleph_2$) + ¬FRP($\aleph_2$) is proved relative to a supercompact cardinal.

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Authors' addresses

Sakaé Fuchino

Chubu University
E-mail address: fuchino@isc.chubu.ac.jp

István Juhász

Alfréd Rényi Institute of Mathematics
E-mail address: juhasz@renyi.hu

Lajos Soukup

Alfréd Rényi Institute of Mathematics
E-mail address: soukup@renyi.hu

Zoltán Szentmiklóssy

Eötvös University of Budapest
E-mail address: zoli@renyi.hu

Toshimichi Usuba

Tohoku University
E-mail address: usuba@math.tohoku.ac.jp