Topological characterizations of some set-theoretic principles

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A metrization theorem

Theorem 1 (S.F., Juhász, Soukup, Szentmiklóssy, Usuba [1])

Assume Fodor-type Reflection Principle (see the next slide). Then the following reflection theorem on metrizability holds:

For a locally countably compact topological space X, if all subspaces Y of X cardinality $\leq \aleph_1$ are metrizable then X itself is metrizable.

[1] S.F., István Juhász, Lajos Soukup, Zoltán Szentmiklóssy and Toshimichi Usuba, *Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness*, Top.and its Appl.,157(8),(2010).

- The reflection theorem on metrizability in [ ] above was proved under Axiom R by Zoltan Balogh (posthumous, 2002).
- $V = L$ refutes the reflection theorem.
- If "locally countably compact" in the assertion above is replaced by "countably compact" then we obtain a theorem in ZFC (Alan Dow's Metrization Theorem (1988)).

Fodor-type Reflection Principle (FRP)

is the following set theoretic principle :

For any regular cardinal $\kappa > \aleph_1$, any stationary $S \subseteq E^{\kappa}_{\omega}$ and $g : S \to [\kappa]^{\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

- $\mathrm{cf}(I) = \omega_1$; $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- for any $f : S \cap I \to \kappa$ s.t. $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

- $\aleph_0$: countable infinite cardinal, $\aleph_1$: the first uncountable cardinal
- A cardinal κ is *regular* if κ is not a union of less than κ many cardinals $< \kappa$ ($\Leftrightarrow$ $\mathrm{cf}(\kappa) = \kappa$)
- $\mathrm{cf}(I)$ the cofinality of the set I of ordinals with the canonical ordering.
- $E^{\kappa}_{\omega} = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$; $[\kappa]^{\lambda} = \{x \subseteq \kappa : |x| = \lambda\}$.
- $S \subseteq \alpha$ is *stationary* if S intersects with all closed unbounded $C \subseteq \alpha$.

Fodor-type Reflection Principle (FRP)

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> - for any $f : S \cap I \to \kappa$ s.t. $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

FRP is a strengthening of the Ordinal Reflection Principle (ORP):

> For any regular cardinal $\kappa > \aleph_1$ and any stationary $S \subseteq E^\kappa_\omega$, there is an $\alpha < \kappa$ with $\mathrm{cf}(\alpha) = \omega_1$, s.t. $S \cap \alpha$ is stationary in α.

with the side condition which reminds of Fodor's Theorem:

> **Fodor's Theorem.** For any regular cardinal κ and $f : \kappa \to \kappa$ s.t. $f(\alpha) < \alpha$ for all $\alpha < \kappa$, there is a $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in κ.

Fodor-type Reflection Principle (FRP)

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Theorem 1 (S.F., Juhász, Soukup, Szentmiklóssy, Usuba [1])

Assume that FRP *holds. Then the following reflection theorem on metrizability holds:*

For a locally countably compact topological space X,

if all subspaces Y of X of cardinality $\leq \aleph_1$ are metrizable then X itself is metrizable.

Fodor-type Reflection Principle (FRP)

is the following set theoretic principle :

For any regular cardinal $\kappa > \aleph_1$, any stationary $S \subseteq E^\kappa_\omega$ and $g : S \to [\kappa]^{\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

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Theorem 2 (S.F. et al.[1]; and S.F., Sakai, Soukup and Usuba [2])

The reflection theorem on metrizability of locally countably compact spaces in Theorem 1 is equivalent to FRP *over* ZFC*.*

[2] S.F., Hiroshi Sakai, Lajos Soukup and Toshimichi Usuba, *More about Fodor-type Reflection Principle*, preprint.

Generalization of Theorem 2

Theorem 2 can be obtained via the proof of the following result:

Theorem 3 (S.F. et al. [1]; and S.F. et al. [2])

FRP *is equivalent to the following assertion:*

> *For a locally separable, countably tight space X, if all subspaces Y of X of cardinality $\leq \aleph_1$ are meta-Lindelöf then X itself is meta-Lindelöf.*

- A space X is *countably tight* if, for any $U \subseteq X$ and $x \in \overline{U}$ there is $U' \in [U]^{\aleph_0}$ s.t. $x \in \overline{U'}$.
- A space X is *meta-Lindelöf* if every open cover of X has a point countable refinement which is also an open cover.

Further results (1/3)

Theorem 4 (S.F. [3]; and S.F. Sakai, Soukup and Usuba [2])

The following assertion is equivalent to FRP*:*

For a T_1 space with point countable base, if all subspaces Y of X of cardinality $\leq \aleph_1$ are left-separated then X itself is left-separated.

[3] S.F., *Left-separated topological spaces under Fodor-type Reflection Principle*, RIMS Kokyuroku No.1619, Combinatorial and Descriptive Set Theory (2008), 32–42.

- A space X is *left-separated* if there is a well-ordering $<$ of X s.t. each initial segment of X w.r.t. $<$ is a closed subset of X.

- W. Fleissner (1986) proved that the assertion of Theorem 4 follows from Axiom R and it is refuted under $\neg$ORP.

Further results (2/3)

Theorem 5 (S.F. et al. [2])

The following assertion is equivalent to FRP*:*

For a countably tight space of local density $\aleph_1$, if all subspaces Y of X of cardinality $\leq \aleph_1$ are collectionwise Hausdorff then X itself is collectionwise Hausdorff.

- A space X is *of local density* κ if for very $p \in X$ there is $Y \in [X]^{\leq \kappa}$ s.t. $p \in int(\overline{Y})$.

- A space X is *collectionwise* Hausdorff if any closed discrete subset D of X can be simultaneously separated by disjoint open sets, i.e., if, for any closed and discrete $D \subseteq X$, there is a family $\mathcal{U}$ of pairwise disjoint open sets such that, for all $d \in D$, there is $U \in \mathcal{U}$ with $D \cap U = \{d\}$.

Further Results (3/3)

- FRP is also equivalent to the reflection theorems in terms of:
  - countable coloring number of infinite graphs (S.F. et al.[2]); and
  - openly generatedness of Boolean algebras (S.F. and Rinot [4]).

[4] S.F. and Assaf Rinot, *Openly generated Boolean algebras and the Fodor-type Reflection Principle*, submitted.

Applications

- The topological (graph-theoretic, Boolean algebraic) reflection principles mentioned above are all equivalent to each other over ZFC.

- All these reflection principles imply Ordinal Reflection Principle.

Theorem 6 (S.F. et al. [1])

FRP *is preserved by c.c.c. generic extension.*

- Hence, all reflection principles above impose almost no restriction on the size of continuum. **cf.:** Under slightly stronger reflection principles, the continuum is $\leq \aleph_2$.

- But these reflection principles (or equivalently FRP) do have certain effect on cardinal arithmetic.

Cardinal Arithmetic under FRP

Theorem 7 (S.F. and Rinot [4])

FRP *implies Shelah's Strong Hypothesis.*

- *Shelah's Strong Hypothesis* (SSH) is the assertion equivalent to the following:

  For every uncountable cardinal κ of countable cofinality, we have $cf([\kappa]^{\aleph_0}, \subseteq) = \kappa^+$.

- By the characterization above of SSH, Singular Cardinal Hypothesis (SCH) follows from SSH.

- Theorem 7 suggests that SSH should be also regarded as a reflection principle. We can in fact characterize SSH in terms of the following topological reflection principle:

SSH as a topological reflection theorem

Theorem 8 (S.F. and Rinot [4])

SSH *is equivalent to the following assertion:*

For any countably tight space X, if X is $< \aleph_1$-thin then X is thin.

- A topological space X is *thin* if $|\overline{D}| \leq |D|^+$ holds for all $D \subseteq X$.
- A topological space X is $<\kappa$-*thin* if $|\overline{D}| \leq |D|^+$ holds for all $D \subseteq X$ of cardinality $< \kappa$.
- In the proof of Theorem 7 and Theorem 8, we use some "heavy" tools from Shelah's Pcf-theory.

Thank you for your attention!

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