Fodor-type Reflection Principle and its "mathematical" characterizations

Sakaé Fuchino

Chubu Univ.

fuchino@isc.chubu.ac.jp
http://pauli.isc.chubu.ac.jp/~fuchino/

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Co-authors and references

The most recent results in this talk are obtained in a joint research with:

Lajos Soukup (Budapest), Hiroshi Sakai (Kobe),
and Toshimichi Usuba (Bonn)

Related Papers and Preprints

[1] S. Fuchino, I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, *Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness*, to appear in Topology and Its Applications.

[2] S. Fuchino, *Left-separated topological spaces under Fodor-type Reflection Principle*, RIMS Kôkyûroku No.1619, (2008), 32–42.

[3] S. Fuchino, *Fodor-type Reflection Principle implies Balogh's theorems under Axiom R*, preprint.

[4] S. Fuchino H. Sakai, L. Soukup and T. Usuba, *More about the Fodor-type Reflection Principle*, in preparation.

$\mathrm{RP}([\lambda]^{\aleph_0})$: For any stationary $S \subseteq [\lambda]^{\aleph_0}$, there is an $I \in [\lambda]^{\aleph_1}$ s.t. $\omega_1 \subseteq I$, $cf(I) = \omega_1$ and $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

$\mathrm{AR}([\lambda]^{\aleph_0})$: For any stationary $S \subseteq [\lambda]^{\aleph_0}$ and ω_1-club $\mathcal{T} \subseteq [\lambda]^{\aleph_1}$, there is $I \in \mathcal{T}$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Here, $\mathcal{T} \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-club (or "tight and unbouded" in Fleissner's terminology) if

- $\mathcal{T}$ is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and
- for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{T}$ of length ω_1, we have $\bigcup_{\alpha<\omega_1} I_\alpha \in T$.

RP $:\Leftrightarrow$ $\mathrm{RP}([\lambda]^{\aleph_0})$ holds for all cardinals $\lambda \geq \aleph_2$.

Axiom R $:\Leftrightarrow$ $\mathrm{AR}([\lambda]^{\aleph_0})$ holds for all cardinals $\lambda \geq \aleph_2$.

$\mathrm{MA}^{+}(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \mathrm{RP}$

Set-theoretic consequences of RP

- (Todorcevic) $2^{\aleph_0} \leq \aleph_2$
- (Foreman, Magidor Shelah) Every poset preserving stationarity of subsets of ω_1 is semiproper. As consequences of this we have e.g.:
- I_{NS} is precipitous
- A strong form of Chang's conjecture
- ...

Mathematical consequences of Axiom R

- Fleissner's Theorem on left-separated spaces
- Fleissner's Theorem on coloring number of graphs
- A characterization of openly generated Bas (F., Qi Feng)
- Balogh's reflection theorem on metrizability
- Balogh's "Theorem 1.4", "Theorem 1.6"
- ...

Fleissner's Theorem on left-separated spaces

Theorem 1 (W. Fleissner 1986)

Assume Axiom R. Suppose that X is a T_1-space with a point countable base. If X is not left-separated then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not left-separated.

- A topological space X is left-separated if there is a well-ordering $<$ of X s.t. every initial segment with respect to $<$ is a closed subset of X.

Fleissner's Theorem on coloring number of graphs

Theorem 2 (W. Fleissner 1986)

Assume Axiom R. If a graph (V, E) *has coloring number* $\geq \aleph_1$ *then there is an infinite subgraph of* (V, E) *of cardinality* $\aleph_1$ *with coloring number* $\aleph_1$*.*

- For a graph (V, E) the coloring number of (V, E) is the minimal cardinal μ s.t.

  there is a well-ordering $\prec$ of V s.t. , for every $v \in V$, the set $\{u \in V \ : \ u \prec v, \{u, v\} \in E\}$ has cardinality $< \mu$.

Balogh's reflection theorem on metrizability

Theorem 3 (Z. Balogh 2002)

Assume Axiom R. Suppose that X *is locally countably compact. If* X *is not metrizable then there is a subspace* Y *of* X *of cardinality* $\leq \aleph_1$ *which is not metrizable.*

- A topological space X is coutably compact if any countable open cover of X has a finite subcover.

- A topological space X is locally countably compact if any point of X has a neighborhood which is countably compact.

Theorem 4 (Z. Balogh 2002 (Theorem 1.4))

Assume Axiom R. Suppose that X is locally Lindelöf, countably tight and s.t.

> *for every subspace Y of X of Lindelöf degree $\leq \aleph_1$ $\overline{Y}$ has Lindelöf degree $\leq \aleph_1$.*

If X is not paracompact then there is a clopen subspace Y of X of Lindelöf degree $\leq \aleph_1$ which is not paracompact.

- A topological space X is Lindelöf if each open cover of X has a countable subcover.
- X is locally Lindelöf if each point of X has a closed neighborhood which is Lindelöf.
- X is countably tight if $x \in \overline{Y}$ for any $x \in X$ and $Y \subseteq X$ then there is a countable $Y' \subseteq Y$ s.t. $x \in \overline{Y'}$.
- X is paracompact if each open covering of X has a locally finite refinement.

Balogh's "Theorem 1.6"

Theorem 5 (Z. Balogh 2002 (Theorem 1.6))

Suppose that X is locally Lindelöf and countably tight. If X is not paracompact then there is an open subspace Y of X of Lindelöf degree $\leq \aleph_1$ which is not paracompact.

- The Lindelöf degree of a topological space X is the minimal cardinal μ s.t. , for any open covering of X there is a subcovering of cardinality $\leq \mu$.

Fodor-type reflection principle (FRP)

The 'mathematical' reflection theorems mentioned above are actually consequences of the following combinatorial principle which is much weaker than Axiom R (even weaker than RP):

$\mathrm{FRP}(\kappa)$: For any stationary $S \subseteq E^{\kappa}_{\omega}$ and mapping $g : S \rightarrow [\kappa]^{\leq\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

- $\mathrm{cf}(I) = \omega_1$;
- $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- for any regressive $f : S \cap I \rightarrow \kappa$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

FRP $:\Leftrightarrow$ $\mathrm{FRP}(\kappa)$ holds for every regular $\kappa \geq \aleph_2$

Fodor-type reflection principle (FRP)

Results from [F., Juhász, Soukup, Szentmiklóssy and Usuba] (and [F.])

- $\mathrm{RP}(\kappa)$ implies $\mathrm{FRP}(\kappa)$ for every regular $\kappa \geq \aleph_2$
- $\mathrm{FRP}(\kappa)$ is preserved by c.c.c. extension (this is of course not the case for $\mathrm{RP}(\kappa)$)

$$\mathrm{MA}^{+}(\sigma\text{-closed}) \;\Rightarrow\; \text{Axiom R} \;\Rightarrow\; \mathrm{RP} \;\Rightarrow\; \mathrm{FRP} \;\Rightarrow\; \mathrm{ORP}$$

- Fleissner's Theorem on left-separated spaces follows from FRP
- The following reflection theorem follows from FRP:
  For a locally countably compact and countably tight space X, if X is not meta-Lindelöf then there is a subspace Y of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf (F, Juhász et al.)
- Balogh's reflection theorem on metrizability follows from FRP
- Balogh's "Theorem 1.6" follows from FRP

Theorem 6 (F., Sakai, Soukup and Usuba)

$\mathrm{FRP} \Leftrightarrow$ *For any regular* $\mu > \aleph_1$*, there is no almost essentially disjoint ladder system* $g : E \to [\mu]^{\aleph_0}$ *for any stationary* $E \subseteq E^\mu_\omega$.

- $E^\mu_\omega = \{\alpha < \mu : \mathrm{cf}(\alpha) = \omega\}$.
- $g : E \to [\mu]^{\aleph_0}$ is a ladder system if, for all $\alpha \in E$, we have $g(\alpha) \subseteq \alpha$, $g(\alpha)$ is cofinal in α and $otp(g(\alpha)) = \omega$.
- A ladder system $g : E \to [\mu]^{\aleph_0}$ is essentially disjoint if there is a regressive $f : E \to \mu$ s.t. $g(\alpha) \setminus f(\alpha)$, $\alpha \in E$ are pairwise disjoint.
- A ladder system $g : E \to [\mu]^{\aleph_0}$ is almost essentially disjoint if $g \restriction X$ is essentially disjoint for all $X \in [\mu]^{<|E|}$.

Theorem 6 (F., Sakai, Soukup and Usuba)

$\mathrm{FRP} \Leftrightarrow$ *For any regular* $\mu > \aleph_1$, *there is no almost essentially disjoint ladder system* $g : E \to [\mu]^{\aleph_0}$ *for any stationary* $E \subseteq E^{\mu}_{\omega}$.

Sketch of the proof. "$\Rightarrow$": Easy.
"$\Leftarrow$": Suppose that $\neg\mathrm{FRP}$ and let λ^* be the least regular cardinal s.t. $\neg\mathrm{FRP}(\lambda^*)$. Then there are a stationary $E \subseteq E^{\lambda^*}_{\omega}$ and a ladder system $g : E \to [\lambda^*]^{\aleph_0}$ s.t. , for any $I \in [\lambda^*]^{\aleph_1}$ with $\mathrm{cf}(I) = \omega_1$ and closed with respect to g, we have:

$$Z_I = \{x \in [I]^{\aleph_0} \ : \ \sup(x) \in E \cap I \text{ and } g(\sup(x)) \subseteq x\}$$

is non-stationary in $[I]^{\aleph_0}$.

Let $\sigma : \lambda^* \to {}^{\aleph_0>}\lambda^*$ be a λ^* to 1 surjection and $C^* = \{\alpha < \lambda^* \ :$ for all $a \in {}^{\aleph_0>}\alpha$, $\{\gamma < \alpha \ : \ \sigma(\gamma) = a\}$ is cofinal in $\alpha\}$. Then C^* is a club subset of λ^*. Thus $E^* = E \cap C^*$ is a stationary subset of λ^*.

Theorem 6 (F., Sakai, Soukup and Usuba)

FRP $\Leftrightarrow$ *For any regular $\mu > \aleph_1$, there is no almost essentially disjoint ladder system $g : E \to [\mu]^{\aleph_0}$ for any stationary $E \subseteq E^{\mu}_{\omega}$.*

For $\alpha \in E^*$, let $\langle \eta^{\alpha}_i : i < \omega \rangle$ be an increasing sequence of ordinals cofinal in α s.t. $\sigma(\eta^{\alpha}_i) = \langle \xi^{\alpha}_k : k \leq i \rangle$ where $\langle \xi^{\alpha}_k : k < \omega \rangle$ is a fixed enumeration of $g(\alpha)$. Let $g^* : E^* \to [\lambda^*]^{\aleph_0}$ be the ladder system defined by $g^*(\alpha) = \{\eta^{\alpha}_i : i < \omega\}$.

g^* is almost essentially disjoint: For $X \in [\lambda^*]^{\aleph_1}$, the essential disjointness of $g^* \restriction X$ can be shown straightfowardly by definition of g^*.

For the essential disjointness of $X \in [\lambda]^{\mu}$ for regular μ with $\aleph_1 < \mu < \lambda^*$, we use $\mathrm{FRP}(\mu)$. □

Using Theorem 6, we can show that most of the reflection theorems mentioned before are actually equivalent to FRP over ZF.

Corollary 7

FRP *is equivalent to each of the following assertions over* ZFC*:*

(A) *For every locally separable countably tight topological space X, if X is not meta-Lindelöf, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not meta-Lindelöf.*

(B) *For every locally countably compact topological space X, if X is not metrizable, then there is a subspaces of X of cardinality $\leq \aleph_1$ which is not metrizable.*

(C) *For every T_1-space X with a point countable base, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(C') *For every metrizable space X, if X is not left-separated, then there is a subspace of X of cardinality $\leq \aleph_1$ which is not left-separated.*

(D) *For any graph $G = \langle G, \mathcal{E} \rangle$ if the coloring number of G is uncountable, then there is $I \in [G]^{\aleph_1}$ s.t. the coloring number of $G \restriction I$ is uncountable.*

Axiom R-like extension of Fodor-type reflection principle (FRP^R)

$\mathrm{FRP}^R(\kappa)$: For any ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, stationary $S \subseteq E^\kappa_\omega$ and mapping $g : S \to [\kappa]^{\leq\aleph_0}$ there is $I \in \mathcal{T}$ such that

- for any regressive $f : S \cap I \to \kappa$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

FRP^R $:\Leftrightarrow$ $\mathrm{FRP}^R(\kappa)$ holds for every regular $\kappa \geq \aleph_2$

- The proof of $\mathrm{RP} \Rightarrow \mathrm{FRP}$ can be modified to prove $\mathrm{Axiom\ R} \Rightarrow \mathrm{FRP}^R$.
- FRP^R is still preserved by c.c.c. extensions.

Axiom R-like extension of Fodor-type reflection principle (FRP^R)

Theorem 8

(A) Assume FRP^R *and "*$\{\kappa < \lambda : \mathrm{cf}([\kappa^{\aleph_0}]) = \kappa\}$ *is cofinal in* λ *for any singular cardinal* λ*" Then, the assertion of Balogh's "Thoerem 1.4" holds.*

(B) The characterization of openly generated BA by F. and Feng holds under FRP^R*.*

Conjecture. FRP^R in the theorem above can be replaced by FRP.

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