Set-theoretic reflection of mathematical properties

Sakaé Fuchino (渕野 昌)

Graduate School of System Informatics
Kobe University
(神戸大学大学院 システム情報学研究科)

http://fuchino.ddo.jp/index-j.html

LC 2017

(2017 年 08 月 22 日 (10:45 JST) version)

2017 年 8 月 18 日 (於 Stockholm University)

This presentation is typeset by pLaTeX with beamer class.
These slides are downloadable as

http://fuchino.ddo.jp/slides/LC2017-reflection-slides-pf.pdf

Background

- Suppose that we have an uncountable structure $\mathfrak{A}$ (possibly in some higher order logic) with a bad property $\mathcal{P}$.

  One of the natural questions:

- Is there a substructure $\mathfrak{B}$ of $\mathfrak{A}$ of smaller cardinality but also with the same bad property $\mathcal{P}$?

  A similar but more general question:

- Suppose that $\mathcal{C}$ is a class of structures and κ is a cardinal. For any $\mathfrak{A} \in \mathcal{C}$, if $\mathfrak{A} \models \mathcal{P}$ for some (bad) property $\mathcal{P}$, is it true that there is always substructures $\mathfrak{B}$ of $\mathfrak{A}$ in $\mathcal{C}$ of cardinality $< \kappa$ with $\mathfrak{B} \models \mathcal{P}$?

- What is the minimal such κ ?

  — We shall call the minimal cardinal κ (or ∞ if there is no such a cardinal κ at all) *the reflection cardinal* of the property $\mathcal{P}$ in the class of structures $\mathcal{C}$.

Non-metrizability of topological spaces

Fact 1. (A. Hajnal and I. Juhász, 1976) *For any uncountable cardinal κ there is a non-metrizable space X s.t. all subspaces Y of X of cardinality $< \kappa$ are metrizable.*

Proof

- Thus, the reflection cardinal of the non-metrizability in all topological spaces is ∞.

Theorem 2. (A. Dow, 1988) *For any compact Hausdorff space X if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable then X is also metrizable.*

- This means that the reflection cardinal of the non-metrizability in compact Hausdorff spaces is $\leq \aleph_2$.
- The compact space $\omega_1 + 1$ with the order topology witnesses that the reflection cardinal is $\geq \aleph_2$.

Non-metrizability of topological spaces (2/3)

- The reflection cardinal of non-metrizability in topological spaces $= \infty$
- The reflection cardinal of non-metrizability in compact Hausdorff spaces $= \aleph_2$

Fact 3. (Folklore ?) *It is consistent that the reflection cardinal of non-metrizability in locally compact Hausdorff spaces is ∞.*

Proof

Theorem 4. ([F., Juhász et al.,2010],
[F., Sakai, Soukup and Usuba])

The statement

> *"the reflection cardinal of non-metrizability in locally compact Hausdorff spaces $= \aleph_2$"*

is consistent modulo a large large cardinal and is equivalent to the Fodor-type Reflection Principle (FRP) over ZFC.

Non-metrizability of topological spaces (3/3)

- The reflection cardinal of non-metrizability in topological spaces $= \infty$
- The reflection cardinal of non-metrizability in compact Hausdorff spaces $= \aleph_2$
- The reflection cardinal of non-metrizability in locally compact Hausdorff spaces can be $\aleph_2$ or ∞, actually can also be many other regular cardinals between them.

▷ The consistency of the statement "The reflection cardinal of non-metrizability in first countable topological spaces is $\aleph_1$" is still open (Hamburger's problem).

Theorem 5. ([Dow, Tall and Weiss, 1990]) *(Assuming the consistency of a supercompact cardinal) the statement*

> *"The reflection cardinal of non-metrizability in first countable topological spaces is $\leq 2^{\aleph_0}$"*

is consistent.

Sketch of a proof

Reflection cardinals of graph coloring

Theorem 6. ([F., Juhász et al.,2010], [F., Sakai, Soukup and Usuba])

The statement

> *"the reflection cardinal of the property [of coloring number $> \aleph_0$] in the class of all graphs $= \aleph_2$"*

is also equivalent to FRP *over* ZFC.

Reflection cardinals of graph coloring (2/3)

- A graph G is called an interval graph if there is a linear ordering $\langle L, <_L\rangle$ s.t. G consists of intervals in L and I, $I' \in G$ are adjacent iff $I \neq I'$ and $I \cap I' \neq \emptyset$.

Theorem 7. ([Todorcevic]) Let κ be a regular cardinal.

The reflection cardinal of the property [of chromatic number $> \kappa$] in the class of interval graphs

$=$ *the reflection cardinal of the property [not κ-special] in the class of trees*

- We denote the reflection cardinal in Theorem 7 by $\mathfrak{Refl}^{\kappa}_{RC}$.
- Rado's Conjecture (RC) is the assertion $\mathfrak{Refl}^{\aleph_0}_{RC} = \aleph_2$.

Reflection cardinals of graph coloring (3/3)

Theorem 8. ([F., Sakai, Torres and Usuba])
The reflection cardinal of the property [of coloring number $> \aleph_0$*] in the class of all graphs* $\leq \mathfrak{Refl}^{\aleph_0}_{RC}$

Corollary 9.
The reflection cardinal of the property [of coloring number $> \aleph_0$*] in the class of all graphs*
$\leq$ *the reflection cardinal of the property*
[of chromatic number $> \aleph_0$*] in the class of all graphs*

Proof. By Theorem 8 and Theorem 7. □

Corollary 10. RC *implies* FRP.

Proof. By Theorem 8 and Theorem 6. □

Generalizations of FRP

The following is a straight forward translation of FRP for cardinals $> \omega$.

- For regular cardinals μ, κ, λ with $\mu^+ < \kappa \leq \lambda$, let
  FRP$(\mu, < \kappa, \lambda)$: For any stationary $S \subseteq E^\lambda_\mu$ and any $g : S \to [\lambda]^\mu$ s.t. $g(\alpha)$ is cofinal in α for all $\alpha \in S$ there is $\gamma < \lambda$ s.t. $\mu < \mathrm{cf}(\gamma) < \kappa$ and the set

$$\{x \in [\gamma]^\mu \ : \ \sup(x) \in S \text{ and } g(\sup(x)) \subseteq x\}$$

  is stationary in $[\gamma]^\mu$.

- ▷ FRP is then the assertion "FRP$(\omega, < \omega_2, \lambda)$ for all regular $\lambda \geq \aleph_2$".

- The straight-forward generalization of FRP by introducing the principle FRP(μ) as "FRP$(\mu, < \mu^{++}, \lambda)$ for all regular $\lambda \geq \mu^{++}$" does not work.

Generalizations of FRP (2/4)

Proposition 11. ([F.,Ottenbreit and Sakai]) For a singular cardinal λ and regular cardinals μ, κ s.t. $\mu^+ < \kappa \leq \lambda$. If all regular $\mu < \nu < \kappa$ satisfies $\forall\delta < \nu$ $(\delta^{\mathrm{cf}(\lambda)} < \nu)$ then $\mathrm{FRP}(\mu, <\kappa, \lambda^+)$ fails.

- For example, under CH, $\mathrm{FRP}(\omega_1, <\omega_{\omega+1}, \omega_{\omega+1})$ fails.

- Note that the following straight-forward generalization of Rado's Conjecture can be easily established by collapsing a strongly compact cardinal:

- For regular cardinals μ, κ, λ with $\mu^+ < \kappa \leq \lambda$, let

  $\mathrm{RC}(\mu, <\kappa, \lambda)$: For any tree T of cardinality λ if T is not μ-special then there is a subtree T' of T of size $<\kappa$ s.t. T' is not μ-special

- For a regular cardinal μ

  $\mathrm{RC}(\mu)$: $\mathrm{RC}(\mu, <\mu^{++}, \lambda)$ holds for all cardinal $\lambda \geq \mu^{++}$.

Generalizations of FRP (3/4)

- The following modification of FRP$(\mu, < \kappa, \lambda)$ will do:
- For regular cardinals μ, κ, λ with $\mu^+ < \kappa \leq \lambda$, let

  FRP$^-(\mu, < \kappa, \lambda)$: For any stationary $S \subseteq E^\lambda_\mu$ and any $g : S \to [\lambda]^\mu$ s.t. $g(\alpha)$ is cofinal in α for all $\alpha \in S$ there is $\gamma < \lambda$ s.t. $\mu < \mathrm{cf}(\gamma) < \kappa$ and the set

  $$\{x \in [\gamma]^\mu \ : \ \sup(x) \in S \text{ and } |\, g(\sup(x)) \cap x \,| = \mu\}$$

  is stationary in $[\gamma]^\mu$.

- For $\mu = \omega$, FRP$(\mu, < \kappa, \lambda)$ coinsides with FRP$^-(\mu, < \kappa, \lambda)$ ([F., Sakai, Soukup and Usuba]).

- For a regular μ,

  FRP$^-(\mu)$: FRP$^-(\mu, < \mu^{++}, \lambda)$ for all regular $\lambda \geq \mu^{++}$.

Generalizations of FRP (4/4)

Theorem 12. ([F.,Ottenbreit and Sakai])*Suppose that μ is a regular cardinal. If $\nu^{<\mu} = \nu$ for all regular cardinal $\nu \geq \mu^{++}$ and $E^{\nu}_{\mu} \in I[\nu^{+}]$ for all singular cardinal $\nu > \mu$ of cofinality $< \mu$. Then* $\mathsf{RC}(\mu)$ *implies* $\mathsf{FRP}^{-}(\mu)$.

Theorem 13 ([F.,Ottenbreit and Sakai]) If κ is a supercompact cardinal and $\mu < \kappa$ is regular then there is a class forcing which preserves cardinals below μ and above κ and which forces $\mathsf{FRP}^{-}(\mu)$.

Theorem 14 ([F.,Ottenbreit and Sakai]) For a regular cardinal the following are equivalent:
(a) $\mathsf{FRP}^{-}(\mu)$,
(b) The reflection cardinal of the property [of coloring number $> \mu$] in the class of all graphs $= \mu^{++}$.

References

- Alan Dow, Franklin D. Tall, William A.R. Weiss, New proofs of the consistency of the normal Moore space conjecture II, Topology and its Applications 37, (1990), 115-129.
- Sakaé Fuchino, István Juhász, Lajos Soukup, Zoltan Szentmiklóssy and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications Vol.157, 8 (2010), 1415–1429.
- Sakaé Fuchino, Hiroshi Sakai, Lajos Soukup and Toshimichi Usuba, More about Fodor-type Reflection Principle, submitted. http://fuchino.ddo.jp/papers/moreFRP.pdf
- Sakaé Fuchino, Hiroshi Sakai, Victor Torres Perez and Toshimichi Usuba, Rado's Conjecture and the Fodor-type Reflection Principle, in preparation. see http://fuchino.ddo.jp/notes/RCimpliesFRP2.pdf

References (2/2)

Sakaé Fuchino, André Ottenbreit Maschio Rodrigues and Hiroshi Sakai, Reflection of properties with uncountable characteristics, in preparation.

Sakaé Fuchino, Pre-Hilbert spaces without orthonormal bases, submitted (https://arxiv.org/pdf/1606.03869v2).

Stevo Todorčević, On a conjecture of Rado, Journal of London Mathematical Society, Vol.S2-27, (1), (1983), 1–8.

Jag tackar för er uppmörksamhet.

御清聴ありがとうございました。

Fodor-type Reflection Principle (FRP)

(FRP) For any regular $\kappa > \omega_1$, any stationary $E \subseteq E^\kappa_\omega$ and any mapping $g : E \to [\kappa]^{\aleph_0}$ with $g(\alpha) \subseteq \alpha$ for all $\alpha \in E$, there is $\gamma \in E^\kappa_{\omega_1}$ s.t.

(*) for any $I \in [\gamma]^{\aleph_1}$ closed w.r.t. g and club in γ, if $\langle I_\alpha : \alpha < \omega_1 \rangle$ is a filtration of I then $\sup(I_\alpha) \in E$ and $g(\sup(I_\alpha)) \subseteq I_\alpha$ hold for stationarily many $\alpha < \omega_1$

▷ $\mathcal{F} = \langle I_\alpha : \alpha < \lambda \rangle$ is a filtration of I if $\mathcal{F}$ is a continuously increasing $\subseteq$-sequence of subsets of I of cardinality $< |I|$ s.t. $I = \bigcup_{\alpha<\lambda} I_\alpha$.

- FRP follows from Martin's Maximum or Rado's Conjecture. $MA^+(\sigma\text{-closed})$ already implies FRP but PFA does not imply FRP since PFA does not imply stationary reflection of subsets of $E^{\omega_2}_\omega$ (Magidor, Beaudoin) which is a consequence of FRP.

- FRP is a large cardinal property: By Fact 3. and Theorem 4., FRP implies the total failure of the square principle.

Back

Proof of Fact 1

Fact 1. (A. Hajnal and I. Juhász, 1976) *For any uncountable cardinal κ there is a non-metrizable space X s.t. all subspaces Y of X of cardinality $< \kappa$ are metrizable.*

Proof.

- Let $\kappa' \geq \kappa$ be of cofinality $\geq \kappa,\ \omega_1$.
- The topological space $(\kappa' + 1, \mathcal{O})$ with

$$\mathcal{O} = \mathcal{P}(\kappa') \cup \{(\kappa' \setminus x) \cup \{\kappa'\} \,:\, x \subseteq \kappa',\ x \text{ is bounded in } \kappa'\}$$

  is non-metrizable since the point κ' has character $\geq \mathrm{cf}(\kappa') > \aleph_0$.
- Any subspace of $\kappa' + 1$ of size $< \kappa$ is discrete and hence metrizable. □

Back

Proof of Fact 3

- It is enough to prove the following:

Lemma. (Folklore ?, see [F., Juhász et al.,2010]) *For a regular cardinal $\kappa \geq \aleph_2$ if, there is a non reflecting stationary $S \subseteq E^{\kappa}_{\omega}$, then there is a non meta-lindelöf (and hence non metrizable) locally compact and locally countable topological space X of cardinality κ s.t. all subspace Y of X of cardinality $< \kappa$ are metrizable.*

Proof.

- Let $I = \{\alpha + 1 : \alpha < \kappa\}$ and $X = S \cup I$.
- Let $\langle a_\alpha : \alpha \in S\rangle$ be s.t. $a_\alpha \in [I \cap \alpha]^{\aleph_0}$, a_α is of order-type ω and cofinal in α. Let $\mathcal{O}$ be the topology on X introduced by letting

  (1) elements of I isolated; and

  (2) $\{a_\alpha \cup \{\alpha\} \setminus \beta : \beta < \alpha\}$ a neighborhood base of each $\alpha \in S$.

- Then $(X, \mathcal{O})$ is not meta-lindelöf (by Fodor's Lemma) but each $\alpha < \kappa$ as subspace of X is metrizable (by induction on α). □ Back

Sketch of the Proof of Theorem 5

Theorem 5. ([Dow, Tall and Weiss, 1990]) *(Assuming the consistency of a supercompact cardinal) the statement*

> *"The reflection cardinal of non-metrizability in first countable topological spaces is $\leq 2^{\aleph_0}$"*

is consistent.

Proof.

- The standard models of real-valued measurability, real-valued Cohenness etc. (i.e. starting from a model with a supercompact cardinal and add that many random (or Cohen) reals etc. (side-by-side)). establish the inequality. □

- The consistency of "The reflection cardinal $= 2^{\aleph_0}$" can be also obtained if we start from a model which satisfies the square principles at cofinally many cardinals below the supercompact κ.

Coloring number and chromatic number of a graph

- For a cardinal $\kappa \in \mathsf{Card}$, a graph $G = \langle G, K \rangle$ has coloring number $\leq \kappa$ if there is a well-ordering $\sqsubseteq$ on G s.t. for all $p \in G$ the set

$$\{q \in G \;:\; q \sqsubseteq p \text{ and } q \; K \; p\}$$

has cardinality $< \kappa$.

もどる

- The coloring number $col(G)$ of a graph G is the minimal cardinal among such κ as above.
- The chromatic number $chr(G)$ of a graph $G = \langle G, K \rangle$ is the minimal cardinal κ s.t. G can be partitioned into κ pieces $G = \bigcup_{\alpha<\kappa} G_\alpha$ s.t. each G_α is pairwise non adjacent (independent).
- For all graph G we have $chr(G) \leq col(G)$.

Tillbaka

κ-special trees

- For a cardinal κ, a tree T is said to be κ-special if T can be represented as a union of κ subsets T_α, $\alpha < \kappa$ s.t. each T_α is an antichain (i.e. pairwise incomparable set).

Back