Openly generated Boolean algebras under FRP

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Openly generated Boolean algebras

- A Boolean algebra B is said to be **openly generated** iff $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club set ($\subseteq [B]^{\aleph_0}$)
- $\mathbf{A} \leq_{\mathrm{rc}} \mathbf{B}$ $\Leftrightarrow$ A is a relatively complete subalgebra of B $\Leftrightarrow$ A is a subalgebra of B and $\forall b \in B$ (the ideal $A \restriction b$ is generated by a single element (lower projection of b)).

Theorem 1 (S.F., Heindorf, Shapiro, 1994)

For a Boolean algebra B, the following are equivalent:

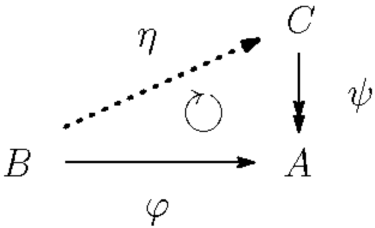
(1) *B is openly generated;*

(2) $\Vdash_{\mathbb{P}}$ "B is projective" *for any σ-closed $\mathbb{P}$ forcing $|B| = \aleph_1$;*

(3) *B has Freese-Nation property. I.e., there is a mapping (Freese-Nation mapping (or FN-mapping)) $f : B \to [B]^{<\aleph_0}$ s.t. $\forall a, b \in B\,(a \leq b \to \exists c \in f(a) \cap f(b)\ (a \leq c \leq b))$.*

Projective Bas

A Boolean algebra B is **projective** $\Leftrightarrow$ for any Boolean mapping $\varphi : B \to A$ and surjective Boolean mapping $\psi : C \to A$ there is a unique Boolean mapping $\eta : B \to C$ s.t.



A Characterization of openly generated Bas

Theorem 2 (S.F. and Q. Feng, 1994)

Assume Axiom R. Then a Boolean algebra B is openly generated if and only if B is $\aleph_2$-projective. $\square$

- B is **$\aleph_2$-projective**

$\Leftrightarrow$ $\{C \in [B]^{<\aleph_2} : C$ is projective$\}$ contains a club ($\subseteq [B]^{<\aleph_2}$)

Theorem 3 (S.F., 1994)

Assume $\square_\kappa$ then there is an $\aleph_2$-projective but not openly generated Boolean algebra B of cardinality κ^+. $\square$

Main Theorem 4 (S.F. and A. Rinot, to appear)

The assertion of Theorem 2:

> *a Boolean algebra B is openly generated if and only if B is $\aleph_2$-projective*

is equivalent to FRP *over* ZFC.

- **FRP** (Fodor-type Reflection Principle) is the following assertion:

For any regular cardinal $\kappa > \aleph_1$, any stationary $S \subseteq E^{\kappa}_{\omega}$ and $g : S \to [\kappa]^{\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

- $\mathrm{cf}(I) = \omega_1$; $g(\alpha) \subseteq I$ for all $\alpha \in S \cap I$;
- for any $f : S \cap I \to \kappa$ s.t. $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

Fodor-type Reflection Principle (1/2)

- **FRP** (Fodor-type Reflection Principle) is the following assertion:

For any regular cardinal $\kappa > \aleph_1$, any stationary $S \subseteq E^\kappa_\omega$ and $g : S \to [\kappa]^{\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

- $\mathrm{cf}(I) = \omega_1$; $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- for any $f : S \cap I \to \kappa$ s.t. $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

- FRP is introduced and studied in [S.F., Juhász, Soukup, Szentmiklóssy and Usuba 2010].

- MM $\Rightarrow$ MA$^+$(σ-closed) $\Rightarrow$ Axiom R $\Rightarrow$ RP $\overset{\not\Leftarrow}{\Rightarrow}$ FRP $\overset{\not\Leftarrow}{\Rightarrow}$ ORP

([S.F. et al. 2010] and [S.F., Sakai, Soukup and Usuba 201?])

Fodor-type Reflection Principle (2/2)

- **FRP** (Fodor-type Reflection Principle) is the following assertion:

For any regular cardinal $\kappa > \aleph_1$, any stationary $S \subseteq E^\kappa_\omega$ and $g : S \to [\kappa]^{\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

- $\mathrm{cf}(I) = \omega_1$; $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- for any $f : S \cap I \to \kappa$ s.t. $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

Lemma 5

FRP *is equivalent to its following variant:*

For any regular cardinal $\kappa > \aleph_1$, *any stationary* $S \subseteq E^\kappa_\omega$ *and* $g : S \to [\kappa]^{\aleph_0}$ *there are stationarily many* $I \in [\kappa]^{\aleph_1}$ *such that*

- $\mathrm{cf}(I) = \omega_1$; $g(\alpha) \subseteq I$ *for all* $\alpha \in I \cap S$;
- $\{x \in [I]^{\aleph_0} : g(\sup(x)) \cap \sup(x) \subseteq x\}$ *is stationary.*

Ingredients of the proof of Main Theorem

Theorem 6 (A. Rinot, independently: T. Usuba)

FRP *implies* SSH.

- SSH (Shelah's Strong Hypothesis) is equivalent to the assertion that $\mathrm{cf}([\lambda]^{\aleph_0}, \subseteq) = \lambda^+$ for all regular cardinal with $\mathrm{cf}(\lambda) = \omega$.

Theorem 7 (S.F.)

Suppose that SSH *holds. Then every* $\aleph_2$*-projective Boolean algebras* B *have a filtration* $\langle B_\alpha : \alpha < \kappa \rangle$ *for* $\kappa = \mathrm{cf}(|B|)$ *s.t.* $B_{\alpha+1}$ *is* $\aleph_2$*-projective and* $B_{\alpha+1} \leq_\sigma B$ *for all* $\alpha < \kappa$*. In particular, we also have* $B_\alpha \leq_\sigma B$ *for all limit* $\alpha < \kappa$ *of countable cofinality.*

- $\mathbf{A} \leq_\sigma \mathbf{B}$ $\Leftrightarrow$ A is a sigma subalgebra of B $\Leftrightarrow$ A is a subalgebra of B and $\forall b \in B$ (the ideal $A \restriction b$ is generated by countable subset of $A \restriction b$).

Sketch of the proof of Main Theorem (1/4)

- "B is openly generated $\Rightarrow$ B is $\aleph_2$-projective" is easy and provable in ZFC.
- FRP $\Rightarrow$ (B is $\aleph_2$-projective $\Rightarrow$ B is openly generated) (*)

By induction on $|\,B\,|$.

▷ If $|\,B\,| \leq \aleph_1$ this is trivial.

▷ Suppose (*) is true for all Ba of cardinality $< |\,B\,| = \lambda$.

Case I: λ is singular: a proof similar to that of Shelah's Singular Compactness Theorem will do.

Case II: λ is regular:

Sketch of the proof of Main Theorem (2/4)

Case II: λ is regular:

Suppose that B is an $\aleph_2$-projective non openly generated Boolean algebra with $|B| = \lambda$.

By Theorem 7, there is a filtration $\langle B_\alpha : \alpha < \lambda \rangle$ of B s.t. $B_\alpha \leq_\sigma B$ for all $\alpha \in \lambda \setminus E^\kappa_{>\omega}$. The filtration can be choosen so that each B_α is $\aleph_2$-projective. By the induction hypothesis, all B_α, $\alpha < \lambda$ are openly generated.

$E = \{\alpha \in E^\lambda_\omega : B_\alpha \leq_{\neg\mathrm{rc}} B\}$ is stationary (otherwise we could conclude that B is openly generated !). Without loss of generality $B_\alpha = \alpha$ for all $\alpha \in E$.

For $\alpha \in E$, there is $\eta_\alpha \in \lambda \setminus \alpha$ s.t. $B_\alpha \restriction \eta_\alpha$ is countably generated by some $c_\alpha \in [B_\alpha \restriction \eta_\alpha]^{\aleph_0}$ but not by any single element. Let $g : E \to [\lambda]^{\aleph_0}$ be defined by $g(\alpha) = c_\alpha \cup \{\eta_\alpha\}$ for $\alpha \in E$.

Sketch of the proof of Main Theorem (3/4)

By the characterization (Lemma 5) of FRP, there is $I \in [\lambda]^{\aleph_1}$ s.t. I is prjective as a subalgebra of B, $\mathrm{cf}(I) = \omega_1$, I is closed w.r.t. g and $\{x \in [I]^{\aleph_0} \ : \ g(\mathrm{sup}(x)) \cap \mathrm{sup}(x) \subseteq x\}$ is stationary.
From the last condition it follows that I is not projective which is a contradiction.

Sketch of the proof of Main Theorem (4/4)

- (B is $\aleph_2$-projective $\Leftrightarrow$ B is openly generated) $\Rightarrow$ FRP:

Assume $\neg$FRP. Then there is a regular $\kappa > \aleph_1$, stationary $S \subseteq E^\kappa_\omega$ and a ladder system $g : S \to [\kappa]^{\aleph_0}$ s.t., for any $\alpha < \kappa$, there is a regressive $f : S \cap \alpha \to \alpha$ s.t. $\{g(\beta) \setminus f(\beta) : \beta \in S \cap \alpha\}$ is pairwise disjoint ([S.F.,Sakai,Soukup and Usuba 201?]).

Let $D = \kappa \setminus Lim$. Without loss of generality, $g(\alpha) \subseteq D$ for all $\alpha \in S$. Let $X = \{c_\alpha : \alpha \in S \cup D\}$.

Let $<_B$ be a binary relation on X defined by

$$c_\alpha <_B c_\beta \;\Leftrightarrow\; \alpha \in D \;\wedge\; \beta \in S \;\wedge\; \alpha \in g(\beta).$$

Let B be the Ba generated over X freely except $<_B$.

▷ Then B is $\aleph_2$-projective but not openly generated.

□ (Main Theorem)

Theorem 6 (A. Rinot, independently: T. Usuba)

FRP *implies* SSH.

(Very Rough) Sketch of the Proof:

Suppose that SSH does not hold. Then there is a better scale $\langle\langle\lambda_i : i<\omega\rangle, \langle f_\alpha : \alpha<\lambda^+\rangle\rangle$ for a cardinal λ with $\mathrm{cf}(\lambda)=\omega$ ([Shelah: Card. Arith.]).

Let $\varphi : {}^{\omega>}\lambda \to \lambda$ be a 1-1 mapping, $E = E^{\lambda^+}_\omega \setminus \lambda$ and let $g : E \to [\lambda^+]^{\aleph_0}$ be s.t. $g(\alpha) = \{\varphi(f_\alpha \restriction n) : n\in\omega\}$.

Then g together with E is a counterexample to $\mathrm{FRP}(\lambda^+)$. □

Thank you for your attention!

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