# On the superuniverse of set-theoretic multiverses

Sakaé Fuchino (渕野 昌)

Graduate School of System Informatics
Kobe University
(神戸大学大学院 システム情報学研究科)

http://kurt.scitec.kobe-u.ac.jp/~fuchino/

(19. September 2015 (14:54 JST) version)

**18. September 2015, 於 京都大学数理解析研究所 (RIMS)**

This presentation is typeset by pLATEX with beamer class.
The printer version of these slides is going to be downloadable as
http://kurt.scitec.kobe-u.ac.jp/~fuchino/slides/RIMS15-multiverses-pf.pdf

## Set-theoretic multiverse

- We consider the multitude of universes (models) of set theory
  (the set-theoretic multiverse).

  Perhaps we can claim that the multitude of universes is the ultimate objective of our set-theoretic research.

- Concerning the foundation of set-theoretic multiverse, the following question seems to be left open:

> Where should we consider all the universes of set theory?

## Universes as countable models

**Setting A:** The outer universe (super-universe) $\Omega$ which accommodates all the universes is a model of $\mathrm{ZFC}$ and the universes are countable transitive models of $\mathrm{ZFC}$ in $\Omega$.

- In this setting, we consider a set-theoretic multiverse as some subclass of the class

$$\mathcal{M} = \{m : m \text{ is a countable transitive model of ZF}\}.$$

**Setting A1:** $\Omega$ is as above. We consider

$$\mathcal{M}_\varepsilon = \{m \in \mathcal{M} : \mathrm{On}^m = \varepsilon\}$$

for some $\varepsilon \in \omega_1$.

## Set-generic multiverses

- For $m$, $m' \in \mathcal{M}$, let $m \mathcal{R} m' \;\Leftrightarrow$

  there are $\varepsilon \in \mathrm{On}$, $n \in \omega$ and $m_0, \ldots, m_{n-1} \in \mathcal{M}_\varepsilon$ s.t. $m_0 = m$, $m_{n-1} = m'$ and one of $m_i$, $m_{i+1}$ is a set-generic extension of the other for all $i < n - 1$.

- Each equivalence class modulo $\mathcal{R}$ makes up a set-generic multiverse.

As what should this be considered?

- As a gedanken-experiment (with toy models of set theory).
- As the "real (super)universe" $\Omega$.
  — We can consider the extension of the axiom system $(\mathrm{ZFC})$ of $\Omega$ by adding assertions like "$\mathcal{M}$ is non-empty", "there is $m \in \mathcal{M}$ satisfying the large cardinal axiom XYZ", "for each set-generic multiverse $\subseteq$ coincides with generic extension" etc.

# Inner models of a model of ZFC

**Observation 1.** If $M$ is an inner model of a model $N$ of $\mathrm{ZFC}$ then there is a poset $\mathbb{P} \in M$ s.t. there is no $(\mathbb{P}, M)$-generic filter in $N$

*Proof.* Let $\mathbb{P} = col(\omega, \omega_1^N)$ or $\mathbb{P} = \mathrm{Fn}(((2^{\aleph_0})^+)^N, 2, < \omega)$, etc. □

**Setting B:**

## A theory of multiverse $\mathrm{MZFC}$



- Let $\mathcal{L}$ be the language $\{\in, set(\cdot), M(\cdot)\}$ where $set(\cdot)$ and $M(\cdot)$ are unary predicate symbol.
- Let $\mathrm{MZFC}$ be the theory consisting of:
  - Extensionality: $\forall X \forall Y\, (\forall Z\, (Z \in X \leftrightarrow Z \in Y) \rightarrow X \equiv Y)$;
  - $\forall X \forall Y\, (X \in Y \rightarrow set(X))$;
  - Pairing: $\forall x \forall y\, (set(x) \wedge set(y) \rightarrow \exists z\, (\text{“}z \equiv \{x, y\}\text{”}))$;
  - Separation: $\forall X_1 \cdots X_n \exists Y\, (Y = \{x : set(x) \wedge \varphi(x, X_1, \ldots, X_n)\})$ for all $\mathcal{L}$-formula $\varphi$ in which only “set quantification” occurs.
  - Union (of sets)
  - $\omega$-Replacement: $\forall F \forall x\, (F \text{ is a class function } \wedge \text{“}x \text{ is a countable set”} \rightarrow$ “the set $\{F(z) : z \in x\}$ exists”$)$
  - Regularity
  - (Global?) Choice

## A theory of multiverse $\mathrm{MZFC}$ (2/2)



- $\forall X\,(M(X) \;\rightarrow\;$ "On $\subseteq X$ and $X$ is transitive");
- $\forall X\,(M(X) \;\rightarrow \varphi^{X})$ for each axiom $\varphi$ of $\mathrm{ZFC}$;
- $M(L)$;
- $\forall X \forall \mathbb{P}\,(M(X) \wedge$ "$\mathbb{P}$ is a po in $X$" $\;\rightarrow$
  $\exists G\,(set(G) \wedge$ "$G$ is an $(M, \mathbb{P})$-generic filter"));
- $\forall X \forall \mathbb{P}\,(M(X) \wedge$ "$\mathbb{P}$ is a po in $X$" $\wedge$
  $set(G) \wedge$ "$G$ is an $(M, \mathbb{P})$ $\;\rightarrow\; \exists Y\,(Y = X[G] \wedge M(Y))$
- $\forall x\,(set(x) \;\rightarrow\; \exists X\,(M(X) \wedge x \in X))$;

**Observation 2.** $\mathcal{R}$ corresponding to the $\mathcal{R}$ in Setting A is definable in $\mathrm{MZFC}$.

- $\forall X \forall Y\,(M(X) \wedge M(Y) \wedge X \mathcal{R} Y \wedge X \subseteq Y \rightarrow$
  "$Y$ is a set generic extension of $X$")

## A model of $\mathrm{MZFC}$



- Suppose that $\kappa$ is an inaccessible cardinal. Let $\mathbb{P} = col(\omega, \kappa)$ and let $G$ be a $(V, \mathbb{P})$-generic filter.
- For all po $\mathbb{Q} \in V_\kappa$, $\mathcal{D} = \{D \in M : D \text{ is a dense subset of } \mathbb{Q}\}$ is countable in $V[G]$ hence there is a $(V, \mathbb{Q})$-generic filter $H$ in $(\mathcal{H}_{\omega_1})^{V[G]}$. Similarly for any intermediate model between $V$ and $V[G]$ obtained by forcing hereditarily of size strictly less than $\kappa$ and pos in them.
- For intermediate models by forcing of size strictly less than $\kappa$, amalgamation property trivially holds.
- By the remarks above, with
  $A =$ definable subsets of $((\mathcal{H}_{\omega_1})^{V[G]}, V_\kappa, \in)$,
  $set^{\mathfrak{A}} = (\mathcal{H}_{\omega_1})^{V[G]}$,
  $M^{\mathfrak{A}} =$ intermediate model between some inner model of $V_\kappa$ and $(\mathcal{H}_{\omega_1})^{V[G]}$ generic over the inner model by a po of size $< \kappa$
  and $\mathfrak{A} = \langle A, set^{\mathfrak{A}}, M^{\mathfrak{A}} \rangle$ we have $\mathfrak{A} \models \mathrm{MZFC}$.

御静聴ありがとうございました.

終