On a/the solution of the Continuum Problem

— Laver-generic large cardinals and the Continuum Problem

Sakaé Fuchino (渕野 昌)

fuchino@diamond.kobe-u.ac.jp

RIMS 2019 workshop: Set Theory and Infinity

2019 年 11 月 23 日 (09:24 JST) 版

The most up-to-date version of the following slides is downloadable as:

http://fuchino.ddo.jp/slides/RIMS19-11-pf.pdf

The results in the following slides ...

are to be found in the following joint papers with André Ottenbereit Maschio Rodriques and Hiroshi Sakai:

[1] Sakaé Fuchino, André Ottenbereit Maschio Rodriques, and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, I, submitted. http://fuchino.ddo.jp/papers/SDLS-x.pdf

[2] ______, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, pre-preprint. http://fuchino.ddo.jp/papers/SDLS-II-x.pdf

[3] ______, Strong downward Löwenheim-Skolem theorems for stationary logics, III — mixed support iteration, in preparation.

[4] ______, Strong downward Löwenheim-Skolem theorems for stationary logics, IV — more on Laver-generically large cardinals, in preparation.

[5] Sakaé Fuchino, and André Ottenbereit Maschio Rodriques, Reflection principles, generic large cardinals, and the Continuum Problem, to appear. http://fuchino.ddo.jp/papers/refl_principles_gen_large_cardinals_continuum_problem-x.pdf

The size of the continuum ...

- is either $\aleph_1$ or $\aleph_2$ or very large!
  - provided that a reasonable strong reflection principle with the reflection number either $\leq \aleph_1$ or $< 2^{\aleph_0}$ should hold.
- The consistency of all of the strong reflection principles involved in the statement above are proved by quite similar arguments.
  - By analysing these proofs, we come to the following:

The size of the continuum ...

- is either $\aleph_1$ or $\aleph_2$ or very large!
- provided that a strong variant of generic large cardinal should exist.

For a class $\mathcal{P}$ of p.o.s, a cardinal κ is a Laver-generically supercompact for $\mathcal{P}$ if, for all regular $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, s.t., for any $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are a inner model $M \subseteq \mathsf{V}[\mathbb{H}]$, and an elementary embedding $j : \mathsf{V} \to M$ s.t.

(1) $crit(j) = \kappa$, $j(\kappa) > \lambda$.

(2) $\mathbb{P}, \mathbb{H} \in M$,

(3) $j''\lambda \in M$.

- κ is Laver-generically superhuge for $\mathcal{P}$ if (3) above is replaced by (3)'' $j''j(\kappa) \in M$.
- κ is Laver-generically super almost-huge for $\mathcal{P}$ if (3) above is replaced by (3)' $j''\delta \in M$ for all $\delta < j(\kappa)$.

The condition $j''\lambda \in M$ vers. ${}^{\lambda}M \subseteq M$

Lemma 1. ([2]) *Suppose that* $\mathbb{G}$ *is a* $(\mathsf{V}, \mathbb{P})$*-generic filter for a p.o.* $\mathbb{P} \in \mathsf{V}$ *and* $j : \mathsf{V} \overset{\prec}{\to} M \subseteq \mathsf{V}[\mathbb{G}]$ *s.t., for cardinals* κ, λ *in* V *with* $\kappa \le \lambda$, *crit*$(j) = \kappa$ *and* $j''\lambda \in M$.

(1) *For any set* $A \in \mathsf{V}$ *with* $\mathsf{V} \models |A| \le \lambda$, *we have* $j''A \in M$.

(2) $j \restriction \lambda$, $j \restriction \lambda^2 \in M$.

(3) *For any* $A \in \mathsf{V}$ *with* $A \subseteq \lambda$ *or* $A \subseteq \lambda^2$ *we have* $A \in M$.

(4) $(\lambda^+)^M \ge (\lambda^+)^{\mathsf{V}}$, *Thus, if* $(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]}$, *then* $(\lambda^+)^M = (\lambda^+)^{\mathsf{V}}$.

(5) $\mathcal{H}(\lambda^+)^{\mathsf{V}} \subseteq M$.

(6) $j \restriction A \in M$ *for all* $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$.

Consistency of Laver-generically supercompact cardinals

Theorem 2. ([2]) (1) *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically supercompact cardinal for σ-closed p.o.s" is consistent as well.*

(2) *Suppose that* ZFC + *"there exists a superhuge cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically super almost-huge cardinal for proper p.o.s" is consistent as well.*

Proof

(3) *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a strongly Laver-generically supercompact cardinal for c.c.c. p.o.s" is consistent as well.*

Proposition 3. ([2]) (1) *Suppose that* κ *is generically measurable by a* ω_1 *preserving* $\mathbb{P}$. *Then* $\kappa > \omega_1$. Proof

(2) *Suppose that* κ *is Laver-generically supercompact for* ω_1*-preserving* $\mathcal{P}$ *with* $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$. *Then* $\kappa = \omega_2$. Proof

(3) *Suppose that* $\mathcal{P}$ *is a class of p.o.s containing a p.o.* $\mathbb{P}$ *s.t. any* $(\mathsf{V}, \mathbb{P})$*-generic filter* $\mathbb{G}$ *codes a new real. If* κ *is a Laver-generically supercompact for* $\mathcal{P}$, *then* $\kappa \leq 2^{\aleph_0}$. Proof

(4) *Suppose that* $\mathcal{P}$ *is a class of p.o.s s.t. elements of* $\mathcal{P}$ *do not add any reals. If* κ *is generically supercompact by* $\mathcal{P}$, *then we have* $2^{\aleph_0} < \kappa$. Proof

(5) *Suppose that* κ *is Laver-generically supercompact for* $\mathcal{P}$ *s.t. all* $\mathbb{P} \in \mathcal{P}$ *are ccc and at least one* $\mathbb{P} \in \mathcal{P}$ *adds a real. Then* $\kappa \leq 2^{\aleph_0}$ *holds and* (a) SCH *holds above* $2^{<\kappa}$. (b) *For all regular* $\lambda \geq \kappa$, *there is a* σ*-saturated normal filter over* $\mathcal{P}_\kappa(\lambda)$. (6) *If* κ *is tightly Laver-generically superhuge for ccc*, *then* $\kappa = 2^{\aleph_0}$.

$+$ -versions of MA

- For a class $\mathcal{P}$ of p.o.s and cardinals μ, κ,

MA$^{+\mu}(\mathcal{P}, <\kappa)$:

For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of $\mathbb{P}$-names s.t. $|\mathcal{S}| \leq \mu$ and $\Vdash_{\mathbb{P}}$"$\underset{\sim}{S}$ is a stationary subset of ω_1" for all $\underset{\sim}{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ s.t. $\underset{\sim}{S}[\mathbb{G}]$ is a stationary subset of ω_1 for all $\underset{\sim}{S} \in \mathcal{S}$.

Theorem 4. ([2]) *For an arbitrary class $\mathcal{P}$ of p.o.s, if $\kappa > \aleph_1$ is a Laver-generically supercompact for $\mathcal{P}$, then MA$^{+\mu}(\mathcal{P}, <\kappa)$ holds for all $\mu < \kappa$.*

The trichotomy

Theorem 5. ([2]) *Suppose that κ is Laver-generically supercompact cardinal for a class $\mathcal{P}$ of p.o.s.*

(A) *If elements of $\mathcal{P}$ are ω_1-preserving and do not add any reals, and $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$, then $\kappa = \aleph_2$ and CH holds. Also, $\mathsf{MA}^{+\aleph_1}(\mathcal{P}, <\aleph_2)$ holds.*

(B) *If elements of $\mathcal{P}$ are ω_1-preserving and contain all proper p.o.s then $\mathsf{PFA}^{+\omega_1}$ holds and $\kappa = 2^{\aleph_0} = \aleph_2$.*

(C) *If elements of $\mathcal{P}$ are μ-cc for some $\mu < \kappa$ and $\mathbb{P}$ contains a p.o. which adds a reals then κ is fairly large and $\kappa \leq 2^{\aleph_0}$ also $\mathsf{MA}^{+\mu}(\mathcal{P}, <\kappa)$. holds for any $\mu < \kappa$.*

Thank you for your attention.

We thank you, Daisuke.

巨大基数は存在する.

Large cardinals exist.

中国 四川省 都江堰景区

Proof of Theorem 2, (2)

Theorem 2, (2) Suppose that ZFC + "there exists a superhuge cardinal" is consistent. Then ZFC + "there exists a Laver-generically super almost-huge cardinal for proper p.o.s" is consistent as well.

Proof. Starting from a model of ZFC with a superhuge cardinal κ, we can obtain models of respective assertions by iterating in countable support with proper p.o.s κ times along a Laver function for super almost-hugeness (see [Corazza]).

- In the resulting model, we obtain Laver-generically super almost-hugeness in terms of proper p.o. $\mathbb{Q}$ in each respective inner model $M[\mathbb{G}]$ of $\mathsf{V}[\mathbb{G}]$. The closedness of M in V in terms of super almost-hugeness implies that $\mathbb{Q}$ is also proper in $\mathsf{V}[\mathbb{G}]$.
- This shows that κ is Laver-generically super almost-huge of proper p.o.s.

もどる

Proof of Proposition 3, (4)

Proposition 3, (4) Suppose that $\mathcal{P}$ is a class of p.o.s s.t. elements of $\mathcal{P}$ do not add any reals. If κ is generically supercompact by $\mathcal{P}$, then we have $2^{\aleph_0} < \kappa$.

Proof. Suppose that $\kappa \leq 2^{\aleph_0}$ and let $\lambda \geq 2^{\aleph_0}$.

- Let $\mathbb{P} \in \mathcal{P}$ be s.t. for some $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ with j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to} M$, $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.
- By elementarity, $M \models$"$j(\kappa) \leq (2^{\aleph_0})^M$". Thus
$$(2^{\aleph_0})^V \geq (2^{\aleph_0})^{V[\mathbb{G}]} \geq (2^{\aleph_0})^M \geq j(\kappa) > \lambda \geq (2^{\aleph_0})^{\mathsf{V}}.$$

This is a contradiction.

もどる

Proof of Proposition 3, (2)

Proposition 3, (2) Suppose that κ is Laver-generically supercompact for ω_1-preserving $\mathcal{P}$ with $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$. Then $\kappa = \omega_2$.

Proof. Suppose that $\kappa \neq \omega_2$. Then, by (1), we have $\kappa > \omega_2$

- Let $\mathbb{Q} \in \mathcal{P}$ be s.t. $\mathbb{P} \lessdot \mathbb{Q}$ for $\mathbb{P} = \mathrm{Col}(\omega_1, \{\omega_2\})$ and s.t., for a $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are M, $j \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \overset{\preceq}{\to} M$, $crit(j) = \kappa$.
- By elementarity, $M \models$" $\underbrace{j((\omega_2)^{\mathsf{V}})}_{=(\omega_2)^{\mathsf{V}}}$ is "ω_2" ". This is a contradiction since $\mathbb{H} \cap \mathbb{P} \in M$ collapes $(\omega_2)^{\mathsf{V}}$ to an ordinal of cardinality $\aleph_1$.

もどる

Proof of Proposition 3, (1)

Proposition 3, (1) Suppose that κ is generically measurable by a ω_1 preserving $\mathbb{P}$. Then $\kappa > \omega_1$.

Proof. Suppose that $\kappa \leq \omega_1$. Since $\kappa = \omega$ is impossible, we have $\kappa = \omega_1$.

- Let $\mathbb{P}$ be an ω_1 preserving p.o. and $\mathbb{G}$ a $(\mathrm{V}, \mathbb{P})$-generic filter with $M, j \subseteq V[\mathbb{G}]$ s.t. $j : \mathrm{V} \overset{\prec}{\to} M$, $crit(j) = \kappa$.
- By elementarity we have $M \models$"$j(\kappa) = \omega_1$".
- Thus $(\omega_1)^V < (\omega_1)^M \leq (\omega_1)^{\mathrm{V}[\mathbb{G}]}$. This is a contradiction to the ω_1 preserving of $\mathbb{P}$.

もどる

Proof of Proposition 3, (3)

Proposition 3, (3) Suppose that $\mathcal{P}$ is a class of p.o.s containing a p.o. $\mathbb{P}$ which adds a new real. If κ is a Laver-generically supercompact for $\mathcal{P}$, then $\kappa \leq 2^{\aleph_0}$.

Proof. Let $\mathbb{P} \in \mathcal{P}$ be s.t. any generic filter over $\mathbb{P}$ codes a new real. Suppose that $\mu < \kappa$. We show that $2^{\aleph_0} > \mu$. Let $\vec{a} = \langle a_\xi : \xi < \mu \rangle$ be a sequence of subsets of ω. It is enough to show that $\vec{a}$ does not enumerate $\mathcal{P}(\omega)$.

- By Laver-generic supercompactness of κ for $\mathcal{P}$, there are $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, transitive $M \subseteq \mathsf{V}[\mathbb{H}]$ and $j \subseteq M[\mathbb{H}]$ with $j : \mathsf{V} \overset{\prec}{\to} M$, $crit(j) = \kappa$ and $\mathbb{P}, \mathbb{H} \in M$. Since $\mu < \kappa$, $j(\vec{a}) = \vec{a}$.
- Since $\mathbb{H} \in M$ where $\mathbb{G} = \mathbb{H} \cap \mathbb{P}$ and $\mathbb{G}$ codes a new real not in V, we have

$$M \models \text{“} j(\vec{a}) \text{ does not enumerate } 2^{\aleph_0}\text{”}.$$

- By elementarity, it follows that

$$\mathsf{V} \models \text{“} \vec{a} \text{ does not enumerate } 2^{\aleph_0}\text{”}.$$

Strong Downward Löwneheim-Skolem Theorem for stationary logic

- $\mathcal{L}^{\aleph_0}_{stat}$ is a weak second order logic with monadic second-order variables X, Y etc. which run over the countable subsets of the underlying set of a structure. The logic has only the weak second order quantifier "*stat*" and its dual "*aa*" (but not the second-order existential (or universal) quantifiers) with the interpretation:

  $\mathfrak{A} \models stat\, X\, \varphi(\ldots, X) \quad :\Leftrightarrow$
  $\{U \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi(\ldots, U)\}$ is a stationary subset of $[A]^{\aleph_0}$.

- For $\mathfrak{B} = \langle B, \ldots\rangle \subseteq \mathfrak{A}$, $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A} \quad :\Leftrightarrow$
  $\mathfrak{B} \models \varphi(a_0, \ldots, U_0, \ldots) \Leftrightarrow \mathfrak{A} \models \varphi(a_0, \ldots, U_0, \ldots)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, \ldots, X_0, \ldots)$ and for all $a_0, \ldots \in B$ and for all $U_0, \ldots \in [B]^{\aleph_0}$.

- SDLS$(\mathcal{L}^{\aleph_0}_{stat}, < \kappa) \quad :\Leftrightarrow$
  For any structure $\mathfrak{A} = \langle A, \ldots\rangle$ of countable signature, there is a structure $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

もどる

A weakening of the Strong Downward Löwneheim-Skolem Theorem

- For $\mathfrak{B} = \langle B, \dots \rangle \subseteq A$, $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$ $\quad :\Leftrightarrow$
  $\mathfrak{B} \models \varphi(a_0, \dots) \Leftrightarrow \mathfrak{A} \models \varphi(a_0, \dots)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, \dots)$ without free seond-order variables and for all $a_0, \dots \in B$.

- $\mathrm{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ $\quad :\Leftrightarrow$
  For any structure $\mathfrak{A} = \langle A, \dots \rangle$ of countable signature, there is a structure $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

もどる

Strong Downward Löwneheim-Skolem Theorem for PKL logic

- $\mathcal{L}^{PKL}_{stat}$ is the weak second-order logic with monadic second order variables X, Y, etc. with built-in unary predicate symbol $\underbrace{K}$. The monadic seond order variables run over elements of $\mathcal{P}_{\underbrace{K}^{\mathfrak{A}}}(A)$ for a structure $\mathfrak{A} = \langle A, \underbrace{K}^{\mathfrak{A}}, \dots\rangle$ where we denote $\mathcal{P}_S(T) = \mathcal{P}_{|S|}(T) = \{u \subseteq T \ : \ |u| < |S|\}$. The logic has the unique second order quantifier "*stat*" (and its dual).

- The internal interpretation of the quantifier is defined by:

  $\mathfrak{A} \models^{int} stat\, X\, \varphi(a_0, \dots, U_0, \dots, X) \quad :\Leftrightarrow$
  $\{U \in \mathcal{P}_{\underbrace{K}^{\mathfrak{A}}}(A) \cap A \ : \ \mathfrak{A} \models^{int} \varphi(a_0, \dots, U_0, \dots, U)\}$ is a stationary subset of $\mathcal{P}_{\underbrace{K}^{\mathfrak{A}}}(A)$ for $a_0, \dots A$ and $U_0, \dots \in \mathcal{P}_{\underbrace{K}^{\mathfrak{A}}}(A) \cap A$.

- For $\mathfrak{B} = \langle B, K \cap B, \dots\rangle \subseteq \mathfrak{A} = \langle A, K, \dots\rangle$, $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A} \quad :\Leftrightarrow$
  $\mathfrak{B} \models^{int} \varphi(a_0, \dots, U_0, \dots) \Leftrightarrow \mathfrak{A} \models^{int} \varphi(a_0, \dots, U_0, \dots)$ for all $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, \dots)$ $a_0, \dots \in B$ and $U_0, \dots \in \mathcal{P}_{K \cap B}(B) \cap B$.

Strong Downward Löwneheim-Skolem Theorem for PKL logic (2/2)

- $\mathrm{SDLS}^{int}(\mathcal{L}^{PKL}_{stat}, <\kappa)$ $:\Leftrightarrow$
  for any regular $\lambda \geq \kappa$ and a structuer $\mathfrak{A} = \langle A, K, \ldots \rangle$ of countable signature with $|A| = \lambda$ and $|K| = \kappa$. $\langle \mathcal{H}(\lambda), \kappa, \in \rangle$, there is a structure $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A}$.

- $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <\kappa)$ $:\Leftrightarrow$
  for any regular $\lambda \geq \kappa$ and a structuer $\mathfrak{A} = \langle A, K, \ldots \rangle$ of countable signature with $|A| = \lambda$ and $|K| = \kappa$. $\langle \mathcal{H}(\lambda), \kappa, \in \rangle$, there are stationarily many structures $\mathfrak{B}$ of size $< \kappa$ s.t. $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A}$.

もどる

tightly Laver generically superhuge cardinals

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is a tightly Laver-generically superhuge for $\mathcal{P}$ if, for all regular $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, s.t., for any $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are a inner model $M \subseteq \mathsf{V}[\mathbb{H}]$, and an elementary embedding $j : \mathsf{V} \to M$ s.t.

  (1) $crit(j) = \kappa$, $j(\kappa) > \lambda$.

  (2) $\mathbb{P}, \mathbb{H} \in M$,

  (3) $j''j(\kappa) \in M$, and

  (4) $|\mathbb{Q}| \leq j(\kappa)$.

Proposition 3. にもどる

もどる

Diagonal Reflection Principle

- (S. Cox) For a regular cardinal $\theta > \aleph_1$:

  DRP(θ, IC): There are stationarily many $M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{\aleph_1}$ s.t.

  (1) $M \cap \mathcal{H}(\theta)$ is internally club;

  (2) for all $R \in M$ s.t. R is a stationary subset of $[\theta]^{\aleph_0}$, $R \cap [\theta \cap M]^{\aleph_0}$ is stationary in $[\theta \cap M]^{\aleph_0}$.

- For a regular cardinal $\lambda > \aleph_1$

  $(*)_\lambda$: For any countable expansion $\tilde{\mathfrak{A}}$ of $\langle \mathcal{H}(\lambda), \in \rangle$, if $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$, is a family of stationary subsets of $[\mathcal{H}(\lambda)]^{\aleph_0}$, then there is an internally club $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and $S_a \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$, for all $a \in M$.

Proposition 1. *TFAE:* (a) *The global version of Diagonal Reflection Principle of S.Cox for internal clubness (i.e.* DRP(θ, IC) *for all regular* $\theta > \aleph_1$*) holds.*

(b) $(*)_\lambda$ *for all regular* $\lambda > \aleph_1$ *holds.*

Diagonal Reflection Principle

- (S. Cox) For a regular cardinal $\theta > \aleph_1$:

  DRP(θ, IC): There are stationarily many $M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{\aleph_1}$ s.t.

  (1) $M \cap \mathcal{H}(\theta)$ is internally club;

  (2) for all $R \in M$ s.t. R is a stationary subset of $[\theta]^{\aleph_0}$, $R \cap [\theta \cap M]^{\aleph_0}$ is stationary in $[\theta \cap M]^{\aleph_0}$.

- For a regular cardinal $\lambda > \aleph_1$

  $(*)_\lambda$: For any countable expansion $\tilde{\mathfrak{A}}$ of $\langle \mathcal{H}(\lambda), \in \rangle$, if $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$, is a family of stationary subsets of $[\mathcal{H}(\lambda)]^{\aleph_0}$, then there is an internally club $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and $S_a \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$, for all $a \in M$.

Proposition 1. *TFAE:* (a) *The global version of Diagonal Reflection Principle of S.Cox for internal clubness (i.e.* DRP(θ, IC) *for all regular* $\theta > \aleph_1$*) holds.*

(b) $(*)_\lambda$ *for all regular* $\lambda > \aleph_1$ *holds.*

(c) $\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *holds.*

Reflection Principles $\mathsf{RP}_{??}$

- The following are variations of the "Reflection Principle" in [Jech, Millennium Book].

$\mathsf{RP}_{\mathsf{IC}}$ For any uncountable cardinal λ, stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and any countable expansion $\mathfrak{A}$ of the structure $\langle \mathcal{H}(\lambda), \in \rangle$, there is an internally club $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. (1) $\mathfrak{A} \restriction M \prec \mathfrak{A}$; and (2) $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

$\mathsf{RP}_{\mathsf{IU}}$ For any uncountable cardinal λ, stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and any countable expansion $\mathfrak{A}$ of the structure $\langle \mathcal{H}(\lambda), \in \rangle$, there is an internally unbounded $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. (1) $\mathfrak{A} \restriction M \prec \mathfrak{A}$; and (2) $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

Since every internally club M is internally unbounded, we have:

Lemma 1. $\mathsf{RP}_{\mathsf{IC}}$ *implies* $\mathsf{RP}_{\mathsf{IU}}$*.*

$\mathsf{RP}_{\mathsf{IU}}$ is also called Axiom R in Set-Theoretic Topology.

Theorem 2. ([Fuchino, Juhasz etal. 2010]) $\mathsf{RP}_{\mathsf{IU}}$ *implies* FRP*.*

もどる

Stationary subsets of $[X]^{\aleph_0}$

- $C \subseteq [X]^{\aleph_0}$ is club in $[X]^{\aleph_0}$ if (1) for every $u \in [X]^{\aleph_0}$, there is $v \in C$ with $u \subseteq v$; and (2) for any countable increasing chain $\mathcal{F}$ in C we have $\bigcup \mathcal{F} \in C$.

▷ $S \subseteq [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$ if $S \cap C \neq \emptyset$ for all club $C \subseteq [X]^{\aleph_0}$.

- A set M is internally unbounded if $M \cap [M]^{\aleph_0}$ is cofinal in $[M]^{\aleph_0}$ (w.r.t. $\subseteq$)

▷ A set M is internally stationary if $M \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$

▷ A set M is internally club if $M \cap [M]^{\aleph_0}$ contains a club in $[M]^{\aleph_0}$.

“Diagonal Reflection Principle” にもどる
“$\mathrm{RP}_{??}$” にもどる

Fodor-type Reflection Principle (FRP)

(FRP) For any regular $\kappa > \omega_1$, any stationary $E \subseteq E^\kappa_\omega$ and any mapping $g : E \to [\kappa]^{\aleph_0}$ with $g(\alpha) \subseteq \alpha$ for all $\alpha \in E$, there is $\gamma \in E^\kappa_{\omega_1}$ s.t.

(*) for any $I \in [\gamma]^{\aleph_1}$ closed w.r.t. g and club in γ, if $\langle I_\alpha : \alpha < \omega_1 \rangle$ is a filtration of I then $\sup(I_\alpha) \in E$ and $g(\sup(I_\alpha)) \subseteq I_\alpha$ hold for stationarily many $\alpha < \omega_1$.

- ▷ $\mathcal{F} = \langle I_\alpha : \alpha < \lambda \rangle$ is a filtration of I if $\mathcal{F}$ is a continuously increasing $\subseteq$-sequence of subsets of I of cardinality $< |\, I \,|$ s.t. $I = \bigcup_{\alpha<\lambda} I_\alpha$.
- ▶ FRP follows from Martin's Maximum or Rado's Conjecture. $\mathrm{MA}^+(\sigma\text{-closed})$ already implies FRP but PFA does not imply FRP since PFA does not imply stationary reflection of subsets of $E^{\omega_2}_\omega$ (Magidor, Beaudoin) which is a consequence of FRP.
- ▶ FRP is a large cardinal property: FRP implies the total failure of the square principle.
- ▷ FRP is known to be equivalent to the reflection of uncountable coloring number of graphs down to cardinality $< \aleph_2$.

もどる

Proof of Fact 1

Fact 1. (A. Hajnal and I. Juhász, 1976) *For any uncountable cardinal κ there is a non-metrizable space X of size κ s.t. all subspaces Y of X of cardinality $< \kappa$ are metrizable.*

Proof.

- Let $\kappa' \geq \kappa$ be of cofinality $\geq \kappa,\ \omega_1$.
- The topological space $(\kappa' + 1, \mathcal{O})$ with

$$\mathcal{O} = \mathcal{P}(\kappa') \cup \{(\kappa' \setminus x) \cup \{\kappa'\} \,:\, x \subseteq \kappa',\ x \text{ is bounded in } \kappa'\}$$

  is non-metrizable since the point κ' has character $= \mathrm{cf}(\kappa') > \aleph_0$.
- Any subspace of $\kappa' + 1$ of size $< \kappa$ is discrete and hence metrizable.

$\square$

もどる

Proof of Fact 3

- It is enough to prove the following:

Lemma 1. (Folklore ?, see [Fuchino, Juhasz etal. 2010]) *For a regular cardinal $\kappa \geq \aleph_2$ if, there is a non-reflectingly stationary $S \subseteq E^\kappa_\omega$, then there is a non meta-lindelöf (and hence non metrizable) locally compact and locally countable topological space X of cardinality κ s.t. all subspace Y of X of cardinality $< \kappa$ are metrizable.*

Proof.

- Let $I = \{\alpha + 1 : \alpha < \kappa\}$ and $X = S \cup I$.
- Let $\langle a_\alpha : \alpha \in S \rangle$ be s.t. $a_\alpha \in [I \cap \alpha]^{\aleph_0}$, a_α is of order-type ω and cofinal in α. Let $\mathcal{O}$ be the topology on X introduced by letting

  (1) elements of I are isolated; and

  (2) $\{a_\alpha \cup \{\alpha\} \setminus \beta : \beta < \alpha\}$ a neighborhood base of each $\alpha \in S$.

- Then $(X, \mathcal{O})$ is not meta-lindelöf (by Fodor's Lemma) but each $\alpha < \kappa$ as subspace of X is metrizable (by induction on α).□ もどる

Coloring number and chromatic number of a graph

- For a cardinal $\kappa \in \mathsf{Card}$, a graph $G = \langle G, K \rangle$ has coloring number $\leq \kappa$ if there is a well-ordering $\sqsubseteq$ on G s.t. for all $p \in G$ the set

$$\{q \in G \,:\, q \sqsubseteq p \text{ and } q \mathrel{K} p\}$$

  has cardinality $< \kappa$.

もどる

- The coloring number $col(G)$ of a graph G is the minimal cardinal among such κ as above.

- The chromatic number $chr(G)$ of a graph $G = \langle G, K \rangle$ is the minimal cardinal κ s.t. G can be partitioned into κ pieces $G = \bigcup_{\alpha<\kappa} G_\alpha$ s.t. each G_α is pairwise non adjacent (independent).

- For all graph G we have $chr(G) \leq col(G)$.

もどる

κ-special trees

- For a cardinal κ, a tree T is said to be κ-special if T can be represented as a union of κ subsets T_α, $\alpha < \kappa$ s.t. each T_α is an antichain (i.e. pairwise incomparable set).

Stationary subset of E^κ_ω

- For a cardinal κ,

$$E^\kappa_\omega = \{\gamma < \kappa : \mathrm{cf}(\gamma) = \omega\}.$$

- A subset $C \subseteq \xi$ of an ordinal ξ of uncountable cofinality, C is closed unbounded (club) in ξ if (1): C is cofinal in ξ (w.r.t. the canonical ordering of ordinals) and (2): for all $\eta < \xi$, if $C \cap \eta$ is cofinal in η then $\eta \in C$.
- $S \subseteq \xi$ is stationary if $S \cap C \neq \emptyset$ for all club $C \subseteq \xi$.
- A stationary $S \subseteq \xi$ if reflectingly stationary if there is some $\eta < \xi$ of uncountable cofinality s.t.$S \cap \eta$ is stationary in η. Thus:
- A stationary $S \subseteq \xi$ if non reflectingly stationary if $S \cap \eta$ is non stationary for all $\eta < \xi$ of uncountable cofinality.

もどる

Proof of Theorem 1.

CH $\Rightarrow$ SDLS($\mathcal{L}^{\aleph_0, II}, < \aleph_2$): For a structure $\mathfrak{A}$ with a countable signature L and underlying set A, let θ be large enough and $\tilde{\mathfrak{A}} = \langle \mathcal{H}(\theta), A, \mathfrak{A}, \in \rangle$. where $A = \underline{A}^{\tilde{\mathfrak{A}}}$ for a unary predicate symbol $\underline{A}$ and $\mathfrak{A} = \underline{\mathfrak{A}}^{\tilde{\mathfrak{A}}}$ for a constant symbol $\underline{\mathfrak{A}}$. Let $\tilde{\mathfrak{B}} \prec \tilde{\mathfrak{A}}$ be s.t. $|\, B \,| = \aleph_1$ for the underlying set B of $\mathfrak{B}$ and $[B]^{\aleph_0} \subseteq B$.
$\mathfrak{B} = \mathfrak{A} \restriction \underline{A}^{\tilde{\mathfrak{B}}}$ is then as desired.

SDLS($\mathcal{L}^{\aleph_0}, < \aleph_2$) $\Rightarrow$ CH: Suppose $\mathfrak{A} = \{\omega_2 \cup [\omega_2]^{\aleph_0}, \in\}$. Consider the $\mathcal{L}^{\aleph_0}$-formula $\varphi(X) = \exists x \forall y\, (y \in x \leftrightarrow y \,\varepsilon\, X)$.
If $\mathfrak{B} = \langle B, ... \rangle$ is s.t. $|\, B \,| \leq \aleph_1$ and $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}}$, then for $C \in [B]^{\aleph_0}$, since $\mathfrak{A} \models \varphi(C)$, we have $\mathfrak{B} \models \varphi(C)$. It dollows that $[B]^{\aleph_0} \subseteq B$ and $2^{\aleph_0} \leq (|\, B \,|)^{\aleph_0} \leq |\, B \,| = \aleph_1$.