Rado's conjecture
and
coloring of graphs

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Seminar in topology and set thoery

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Reflection principles and refleciton numbers

- We consider the following type of statenemts of reflection of properties $\mathcal{P}$:

> For any structure A in the class $\mathcal{C}$, if the property $\mathcal{P}$ holds in A, then there is a substructure $B \in \mathcal{C}$ of A of cardinality $< \kappa$ s.t. $\mathcal{P}$ holds already in B.

- We can also consider the reflection as a compactness statement on $\neg\mathcal{P}$:

> For any structure A in the class $\mathcal{C}$, if the property $\neg\mathcal{P}$ holds in all subustructures $B \in \mathcal{C}$ of A of cardinality $< \kappa$ then $\neg\mathcal{P}$ also holds in A.

- The smallest cardinal κ with the property above is denoted by $\mathfrak{Refl}_{\mathcal{P}}(\mathcal{C})$ or simply by $\mathfrak{Refl}_{\mathcal{P}}$ if $\mathcal{C}$ is clear from the context.

- ▷ The following slides are update of the slides I used for a talk in Vienna on 22 June 2012.

Reflection numbers

- $\mathfrak{Refl}_{\mathrm{Rado}}$ = the minimal κ s.t., for any tree T, if every subset of size $< \kappa$ is special then T is also special

  = the minimal κ s.t., for any linear ordering L and any family $\mathcal{A}$ of invervals in L, if any subgraph $\langle \mathcal{B}, I_{\mathcal{B}} \rangle$ of intersection graph $\langle \mathcal{A}, I_{\mathcal{A}} \rangle$ of size $< \kappa$ is of countable chromatic number, then $\langle \mathcal{A}, I_{\mathcal{A}} \rangle$ also is of countable chromatic number (S.Todorčević).

- A subset T' of a tree T is **special** if T' is a union of countably many subsets T'_n, $n \in \omega$ s.t. each T'_n is pairwise incomparable ($\Leftrightarrow \exists$ strictly order preserving $f : T' \to \mathbb{Q}$ (D. Kurepa)).

- For x, $y \in \mathcal{A}$, $x \; \mathbf{I}_{\mathcal{A}} \; y$ if and only if $x \neq y$ and $x \cap y \neq \emptyset$.

Rado conjecture (RC) $\Leftrightarrow$ $\mathfrak{Refl}_{\mathrm{Rado}} = \aleph_2$.

Results in this talk are going to be included in a joint paper in preparation with:

Hiroshi Sakai, Victor Torres and Toshimichi Usuba.

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- $\mathfrak{Refl}_{\text{FRP}}$ = the minimal κ s.t., for any regular $\lambda \geq \kappa$, stationary $E \subseteq E^{\lambda}_{\omega}$ and for any ladder system $g : E \to [\lambda]^{\aleph_0}$, there is $\alpha \in E^{\lambda}_{\geq\omega_1} \cap E^{\lambda}_{<\kappa}$ s.t. $\{x \in [\alpha]^{\aleph_0} : \sup(x) \in E, g(\sup(x)) \subseteq x\}$ is stationary in $[\alpha]^{\aleph_0}$. Notation and Definitions

Rado conjecture (RC) $\Leftrightarrow$ $\mathfrak{Refl}_{\text{Rado}} = \aleph_2$.

Fodor-type Reflection Principle (FRP) $\Leftrightarrow$ $\mathfrak{Refl}_{\text{FRP}} = \aleph_2$.

▷ RC and FRP are consistent with ZFC (under some large cardinal $\lesssim$ a strongly compact cardinal).

Reflection numbers (2/3)

- $\mathfrak{Refl}_{chr}$ = the minimal κ s.t., for any graph G, if $chr(H) \leq \omega$ for all $H \in [G]^{<\kappa}$ then $chr(G) \leq \omega$.
- $\mathfrak{Refl}_{col}$ = the minimal κ s.t., for any graph G, if $col(H) \leq \omega$ for all $H \in [G]^{<\kappa}$ then $col(G) \leq \omega$.

▷ For a graph $G = \langle G, E \rangle$,
$col(\mathbf{G})$ = the minimal cardinal κ s.t. there is a well-ordering $\sqsubset$ of G with the property that $|\{y \in G : y \sqsubset x, x\ E\ y\}| < \kappa$ for all $x \in G$.

- $\mathfrak{Refl}_{\text{Rado}} \leq \mathfrak{Refl}_{chr}$ (by Todorčević's characteriziation of $\mathfrak{Refl}_{\text{Rado}}$).
- $\beth_\omega \leq \mathfrak{Refl}_{chr} \leq$ ω_1-strongly compact cardinal.

  (Erdős and Hajnal, 1968? + ??).
- $\aleph_1 < \mathfrak{Refl}_{col} = \mathfrak{Refl}_{\text{FRP}}$

  (to appear in [S.F., H.Sakai, V.Torres and T.Usuba]).
- $\mathfrak{Refl}_{col} = \infty$ is possible (this holds e.g. under $V = L$).

Reflection numbers (3/3)

Theorem. (to appear in [S.F., H. Sakai, V. Torres and T. Usuba])
$\mathfrak{Refl}_{\mathrm{FRP}} \leq \mathfrak{Refl}_{\mathrm{Rado}}$.

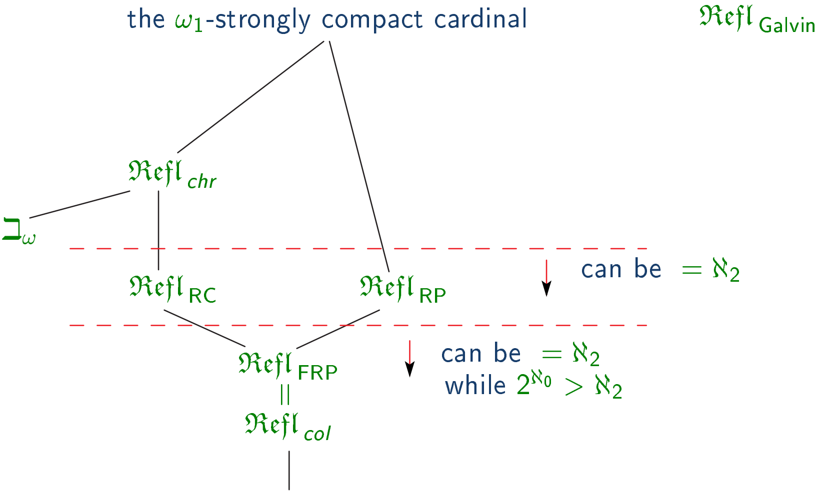
Corollary. (1) Rado Conjecture implies Fodor-type Reflection Principle.

(2) $\mathfrak{Refl}_{col} \leq \mathfrak{Refl}_{chr}$.

Questions.

- $\mathfrak{Refl}_{col} \leq \mathfrak{Refl}_{list\text{-}chr} \leq \mathfrak{Refl}_{chr}$? **No!**:
  $\mathfrak{Refl}_{list\text{-}chr} < \mathfrak{Refl}_{col}$ is consistent (T. Usuba).
  (It is still open if $\mathfrak{Refl}_{list\text{-}chr} \not\leq \mathfrak{Refl}_{chr}$ is consistent with ZFC)
- Is it consistent that $\mathfrak{Refl}_{\mathrm{FRP}} < \mathfrak{Refl}_{\mathrm{Rado}} = \infty$?
  ($\mathfrak{Refl}_{\mathrm{FRP}} < \mathfrak{Refl}_{\mathrm{Rado}}$ is known to be consistent)
- Is Galvin conjecture consistent ?

Reflection numbers



$\mathfrak{Refl}_{\text{RP}}$ = the minimal κ s.t. for any regular cardinal λ and stationary $S \subseteq [\lambda]^{\aleph_0}$ there is $\alpha \in E^\lambda_{\geq\omega_1} \cap E^\lambda_{<\kappa}$ with the property that $S \cap [\alpha]^{\aleph_0}$ is stationary in $[\alpha]^{\aleph_0}$.

Proof of the Theorem

Theorem. (to appear in [S.F., H. Sakai, V. Torres and T. Usuba])
$\mathfrak{Refl}_{\mathsf{FRP}} \leq \mathfrak{Refl}_{\mathsf{Rado}}$.

Proof. Suppose that κ is a regular cardinal $< \mathfrak{Refl}_{\mathsf{FRP}}$. It is enough to show that $\kappa < \mathfrak{Refl}_{\mathsf{Rado}}$ — note that $\mathfrak{Refl}_{\mathsf{FRP}}$ cannot be a successor of a singular cardinal by definition.

- By the assumption, there is a regular λ, stationary $E \subseteq E^{\lambda}_{\omega}$ and a ladder system $g : E \to [\lambda]^{\aleph_0}$ s.t.

$$S = \{x \in [\lambda]^{\aleph_0} \ : \ \sup(x) \notin x,\ g(\sup(x)) \subseteq x\}$$

is stationary (this is always the case) but

$$S_\alpha = \{x \in [\alpha]^{\aleph_0} \ : \ \sup(x) \notin x,\ g(\sup(x)) \subseteq x\}$$

for all $\alpha \in E^{\lambda}_{\geq\omega_1} \cap E^{\lambda}_{<\kappa}$ is non-stationary.
- For $x, y \in S$, let $x \prec y :\Leftrightarrow x \subseteq y$ and $\sup(x) < \sup(y)$.

Proof of the Theorem

$$S = \{x \in [\lambda]^{\aleph_0} : \sup(x) \notin x, \, g(\sup(x)) \subseteq x\}$$

- For $x, y \in S$, let $x \prec y :\Leftrightarrow x \subseteq y$ and $\sup(x) < \sup(y)$.
- Let **T** be the set of all continuously $\prec$-increasing sequence $t = \langle x_\alpha : \alpha < \delta\rangle$ in S of length $< \omega_1$ s.t. $\bigcup_{\alpha<\delta} x_\alpha \in S$.
- For $t, t' \in T$ let $t <_{\mathbf{T}} t'$ if t' is an end-extension of t.

- $T = \langle T, <_T\rangle$ above witnesses $\kappa < \mathfrak{Refl}_{\text{Rado}}$:

Claim.
(1) T is not special.
(2) For every $X \in [S]^{<\kappa}$, $T^X = \{t \in T : \bigcup t \subseteq X\}$ is special.

(1): Since T is a Baire tree, it is not special.

(2): By induction on $otp(X)$.

□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Chromatic number of a graph

- For a graph $G = \langle G, E \rangle$ the **chromatic number** $chr(G)$ of G is the minimal cardinal κ s.t. there is a mapping (coloring) $f : G \to \kappa$ s.t., for any adjacent x, $y \in G$, we have always $f(x) \neq f(y)$.

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Ladder system etc.

- $\mathfrak{Refl}_{\mathsf{FRP}}$ = the minimal κ s.t., for any regular $\lambda \geq \kappa$, stationary $E \subseteq E^{\lambda}_{\omega}$ and for any ladder system $g : E \to [\lambda]^{\aleph_0}$, there is $\alpha \in E^{\lambda}_{\geq\omega_1} \cap E^{\lambda}_{<\kappa}$ s.t. $\{x \in [\alpha]^{\aleph_0} \ : \ \sup(x) \in E, \ g(\sup(x)) \subseteq x\}$ is stationary in $[\alpha]^{\aleph_0}$.

- For a cardinal λ and a regular cardinal $\kappa < \lambda$

  $\mathbf{E}^{\lambda}_{\kappa} = \{\alpha \in \lambda \ : \ \mathrm{cf}(\alpha) = \kappa\}$;

  $\mathbf{E}^{\lambda}_{<\kappa}$, $\mathbf{E}^{\lambda}_{\geq\kappa}$ etc. are defined similarly.

- For a set X and a cardinal κ

  $[\mathbf{X}]^{\kappa} = \{x \subseteq X \ : \ |x| = \kappa\}$.

  $[\mathbf{X}]^{<\kappa}$, $[\mathbf{X}]^{\geq\kappa}$ etc. are defined similarly.

- For $E \subseteq E^{\lambda}_{\omega}$, $g : E \to [\lambda]^{\aleph_0}$ is a **ladder system** if $f(\alpha)$ is a cofinal subset of α of cofinality ω for all $\alpha \in E$.

back

ω_1-strongly compact cardinal

- A cardinal κ is **ω_1-strongly compact** if it is the smallest κ with the property that, for any $\mathcal{L}_{\omega_1,\omega}$ theory T, if all subtheories of T of cardinality $< \kappa$ are satisfiable (i.e. T is $< \kappa$-satisfiable) then T itself is satisfiable.

- ▷ $\mathcal{L}_{\omega_1,\omega}$ is the logic defined similarly to the first order logic but additionally it is also allowed to build a conjunction or disjunction of $< \omega_1$ many formulas.

- Cf.: A cardinal κ is **strongly compact** if, for any $\mathcal{L}_{\kappa,\kappa}$ theory T, if all subtheories of T of cardinality $< \kappa$ are satisfiable (i.e. T is $< \kappa$-satisfiable) then T itself is satisfiable.

- ▷ $\mathcal{L}_{\kappa,\lambda}$ is the logic defined similarly to the first order logic but addicitonally it is also allowed to build a conjunction or disjunction of $< \kappa$ many formulas as well as quantification over a block of $< \lambda$ many formulas.

back

List chromatic number of a graph

- For a graph $G = \langle G, E \rangle$, the **list-chromatic number** of G is defined by:

$$\textbf{list-chr(G)} = \min\{\kappa \in \mathrm{Card} : \text{for } \mu = |\, G \,| \text{ and for any } l : G \to [\mu]^{\kappa} \text{ there is a good coloring } f : G \to \mu \text{ s.t. } f(x) \in l(x) \text{ for all } x \in G\}.$$

- $\mathfrak{Refl}_{\textit{list-chr}}$ = the minimal κ s.t. for any graph G, if every subgraph of G of size $< \kappa$ has countable list-chromatic number then G also has countable list-chromatic number.

back

Galvin's conjecture

- $\mathfrak{Refl}_{\text{Galvin}} =$ the minimal κ s.t. for all partial ordering P, if every subordering Q of cardinality $< \kappa$ are union of countably many linear ordered sets, P is also a union of countably many linear ordered sets.
- **Galvin's conjecture** (GC) $\Leftrightarrow$ $\mathfrak{Refl}_{\text{Galvin}} = \aleph_2$.
- We have

$$\mathfrak{Refl}_{\text{Rado}} \leq \mathfrak{Refl}_{\text{Galvin}} \leq \mathfrak{Refl}_{chr}.$$

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