Mathematics seen from anecdotes

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Mathematics seen from anecdotes

- What is mathematics ?
- Mathematical Proof
- Uselessness of mathematics
- Usefulness of mathematics
- Ultimate objective(?) of mathematics

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What is mathematics ?

- In this talk, we want to discuss about what a mathematician would call mathematics.
  - The mathematics in this sense is perhaps connected to but not identical with the school mathematics. It is even quite different from, for example, what a physicist would call mathematics.
- Mathematics for mathematicians is not only the study of what have been already established as mathematical theories; Mathematics for them is primarily the creation of new (mathematical) theories.
  - **A mathematician proves (new) theorems** (that is, true mathematical statements) to obtain a better understanding of (the idealized world consisting of) mathematical objects.

> Ein Mathematiker ist eine Maschine,
> die Kaffee in Sätze verwandelt.
>
> **(A mathematician is a machine for turning coffee into theorems.)** — Alfréd Rényi (1921 – 1970).

What is mathematics ?

- To find a proof of a mathematical conjecture you very often need a very pure form of **imagination** and even fantasy.

- The following citation is from the famous biography of **David Hilbert** (1862–1943) by Constance Reid (1918–2010):

> "It seems that there was a mathematician who had become a novelist. 'Why did he do that?' people in Göttingen marvelled. 'How can a man who was a mathematician write novels?'
>
> '**But that is completely simple**,' Hilbert said. '**He did not have enough imagination for mathematics, but he had enough for novels.**' "
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> — Constance Reid, "Hilbert", (1969).

What is mathematics ?

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David Hilbert is considered to be one of the two best mathematicians in the world at the turn of the 19/20th century, the other best mathematician being Henry Poincaré. Hilbert made the mathematical department of University of Göttingen the world center of mathematical research.
The picture of Hilbert on the right was on the post cards sold at the university as a souvenir to the visitors from all over the world.

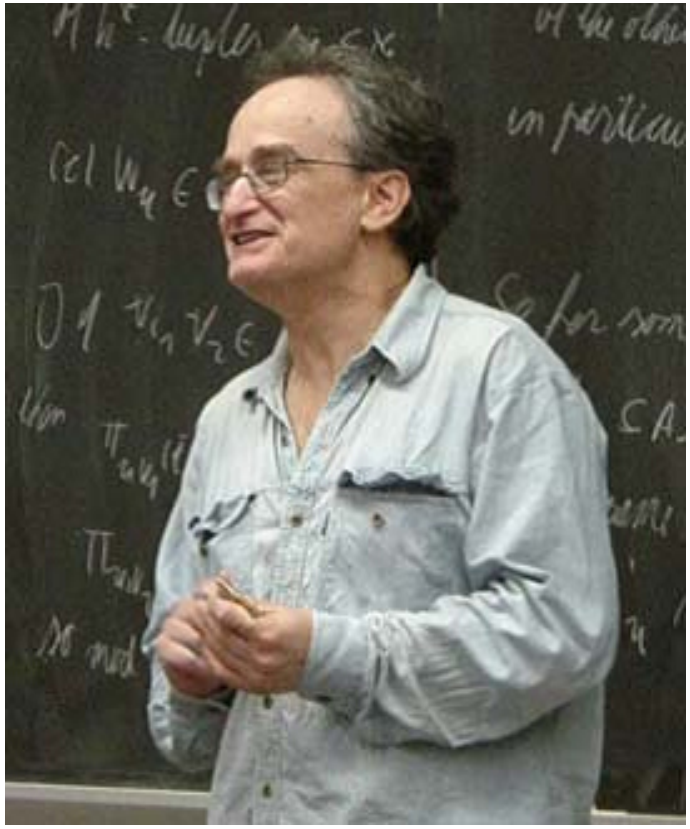
What is mathematics ?

- For many mathematicians, beauty is the most important factor of mathematics:

Interviewer: If you were to explain to a six-year-old child what it is that you do, what would you say?

Saharon Shelah: I would tell them I am making mental constructions which are exact and beautiful.

— Franklin D. Tall and Ann Jane Grieve, Some psychological speculations concerning the modus operandi of Saharon Shelah.



Saharon Shelah (1945 –) surely is one of the most genius mathematicians in the whole history of mathematics.

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Mathematical Proof

- Proofs are centerpiece of mathematics.
- Ancient Greeks introduced the notion of mathematical proof.

> Hippasus of Metapontum (5th century BCE), was a Pythagorean philosopher. Little is known about his life or his beliefs, but he is sometimes credited with the discovery of the existence of irrational numbers. The discovery of irrational numbers is said to have been shocking to the Pythagoreans, and Hippasus is supposed to have drowned at sea, apparently as a punishment from the gods for divulging [leaking, revealing] this. ...
> Some modern scholars though have suggested that he discovered the irrationality of $\sqrt{2}$, which is believed to have been discovered around the time that he lived.
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> — "Hippasus" in Wikipedia ([...] is added by S.F.)

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Pythagoras founded his school, the Pythagoreans, at Croton c. 530 BCE. One of the central dogmas of Pythagoreans was that the world is made from (natural) numbers. Thus, for them, all the numbers appearing as ratios should be rational numbers.

Mathematical Proof

- $\sqrt{2}$ is the positive number a whose square is 2 (that is, $a^2 = 2$ holds). ▶ A number a is rational if a can be represented as a quotient $\frac{m}{n}$ where m is an integer (a whole number) and n is a non-zero natural number (a counting number). A number which is not rational is called an irrational number.

Theorem 1. $\sqrt{2}$ is not a rational number. That is, $\sqrt{2}$ is irrational.

Proof. Suppose that $\sqrt{2}$ were a rational number. That is, it could be represented as $\frac{m}{n}$. We may assume that **the fraction $\frac{m}{n}$ is reduced.** Since $\sqrt{2} = \frac{m}{n}$, We have $2 = \frac{m^2}{n^2}$ or $2n^2 = m^2$. It follows that m must be an even number: let $m = 2m_0$. Substituting this to the equation above we obtain $2n^2 = 4(m_0)^2$ or $n^2 = 2(m_0)^2$. It follows that n also must be an even number: let $n = 2n_0$. But then we can reduce $\frac{m}{n}$ to $\frac{m_0}{n_0}$. This is **a contradiction** since we chose $\frac{m}{n}$ to be reduced! □

Exercise. $\sqrt{3}$ is also irrational. For which n is $\sqrt{n}$ irrational?

Mathematical Proof

Exercise. $\sqrt{3}$ is also irrational. For which n is $\sqrt{n}$ irrational?

- Exercises in mathematics mean that you are asked to find the exact statement of the assertions and their proofs.
- Thus the real question of the exercise above is :

  Give a proof of the statement that $\sqrt{3}$ is irrational. State a theorem giving the exact condition when the natural number n for which $\sqrt{n}$ is irrational, and prove this theorem.

Mathematical Proof

- Theorem 1 on the last slide leads to many natural observations and new questions. The following are among them:

- There are two different irrational numbers whose sum is a rational number. For example: $(1+\sqrt{2})+(1-\sqrt{2})=2$.
- There are two different irrational numbers whose product is a rational number. For example: $(1+\sqrt{2})(1-\sqrt{2})=1^2-(\sqrt{2})^2=1-2=-1$.

Question 1. Are there two different irrational numbers a, b such that a^b is rational?

Mathematical Proof

- The answer to Question 1. is "Yes".

Theorem 2. There are two different irrational numbers a, b s.t. a^b is a rational number.

Proof. If $(\sqrt{3})^{\sqrt{2}}$ is rational then $a = \sqrt{3}$ and $b = \sqrt{2}$ are as desired.
If $(\sqrt{3})^{\sqrt{2}}$ is irrational then, since
$\left((\sqrt{3})^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{3})^{\sqrt{2}\cdot\sqrt{2}} = (\sqrt{3})^2 = 3$, $a = (\sqrt{3})^{\sqrt{2}}$ and $b = \sqrt{2}$ is as desired. □

- Actually Gelfond's Theorem (proved in 1930's) implies that $(\sqrt{3})^{\sqrt{2}}$ is irrational. The proof above however shows the existence of the pair of numbers with the desired property without using this heavy theorem. ▶ In exchange, the present proof does not say how the pair (a, b) looks like. The proof says merely they are either the pair $(\sqrt{3}, \sqrt{2})$, or the pair $((\sqrt{3})^{\sqrt{2}}, \sqrt{2})$.

Coffee Break

Paul Erdős (1913–1996) doing mathematics at the Mathematical Institute Oberwolfach in 1993. A coffee cup is visible in the picture. In his popular talks Erdős mentioned so often the joke with coffee of Rényi that it is often misinterpreted that it were his joke.

Alfréd Rényi (1921–1970). The institute of mathematics of Hungarian academy of science in now called after him.

Mathematical Proof

- The following chain of remarks and questions also arise naturally from Theorem 1:

  - $\frac{m}{n}$ is the unique solution of the equation $nx = m$.
  - $\sqrt{2}$ is one of the two solutions of the equation $x^2 = 2$ (another solution of this equation is $-\sqrt{2}$).

Question 2. Are there numbers for which there is no such (algebraic) equation?

Numbers as in Question 2 are called transcendental numbers. By the first remark above, all transcendental numbers are irrational numbers. A number which is not transcendental, that is, which is one of the solutions of an algebraic equation is called algebraic numbers.

Mathematical Proof

- Several (positive) answers to Question 2. are known:

  - There are transcendental numbers because π is such a number. (F. von Lindemann, 1882)
  - There are as many transcendental numbers as all numbers (G. Cantor, 1874).

Mathematical Proof

- A mathematical proof can be non-constructive.
  That is, it can prove the existence of an object of a certain property without any concrete description or of the object or without describing how the object can be obtained.

  - The non-constructivity makes mathematics a much more powerful theory than mere calculation.

Mathematical Proof

- Mathematics can develop autonomously without any reference to the world outside the mathematics.
  - For example, we could follow chains of remarks and natural questions to arrive Theorem 2. and answers of Question 2. starting from Theorem 1 without any reference to the physical world outside mathematics.

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Uselessness of mathematics

- There are people who maintain that mathematics is useless.
- There are also mathematicians who claim that mathematics is useless.
- But they often claim that mathematics important just because they are useless:

> Real mathematics has no effects on war. No one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years. ... So a real mathematician has his conscience clear; there is nothing to be set against any value his work may have; mathematics is, as I said at Oxford, a 'harmless and innocent' occupation.
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> — G.H. Hardy (1877 – 1947)
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> A Mathematician's Apology (1940)

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G.H. Hardy was one of the British mathematicians who represents the British mathematics of the beginning of the 20th century. He is also remembered as the mathematician who "discovered" the genius Indian mathematician Srinivasa Ramanujan (1887–1920).

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Usefulness of mathematics

- The theories of numbers and relativity Hardy mentioned as example of useless 'real' mathematics are now known to be extremely useful and even crucial to war technology:

  - Number Theory provides the basics of modern cryptography.

  - The equation $e = mc^2$, a part of the Special Theory of Relativity, is the theoretical foundation of atomic energy and hence the basic for the technology of atomic bombs!

Usefulness of mathematics

- The relation of the 'useless' pure mathematics to its applications is well described in the historical radio speech of 1930 by **David Hilbert** (1862 – 1943):

> Without mathematics today's astronomy and physics would be impossible; in their theoretical parts, these sciences unfold directly into mathematics. These, like numerous other applications, give mathematics whatever authority it enjoys with the general public. Nevertheless, all mathematicians have refused to let applications serve as the standard of value for mathematics. ...
> With astonishing sharpness, the great mathematician POINCARÉ once attacked TOLSTOY, who had suggested that pursuing "science for science's sake" is foolish. The achievements of industry, for example, would never have seen the light of day had the practical-minded existed alone and had not these advances been pursued by disinterested fools.
>
> — translated from German by James T. Smith

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Ultimate objective(?) of mathematics

- The German mathematician Carl Gustav Jacob Jacobi (1804 – 1851) wrote in 1830 in a letter to Legendre:

> M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels ; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain, et que sous ce titre, une question de nombres vaut autant qu'une question du système du monde.

Mr. Fourier has the opinion that the main objective of the mathematics is public utility and explanation of natural phenomenons; but a philosopher like him should have known that the unique objective of science is the honor of the human spirit, and in this respect a problem about (natural) numbers is as valuable as a question in many-body problem.

— translation by S.F.

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- Is Mathematics politically correct ?

Is mathematics politically correct?

- All the mathematicians I mentioned in this talk have been male Europeans.
- In the period of 17th century CA to the early 20th century CA, Europa was the only place where modern mathematics was studied. Mathematics in North America in international level did not exist till late 19th century. In Japan European mathematics almost did not exist before the Meiji Restoration (in the middle of 19th century). Teiji Takagi (1875 – 1960) is considered to be the first Japanese mathematician of international level.
- Thus it is inevitable that all the mathematician I mentioned in these slides are from Europe.
- Europa around this time was very conservative concerning the gender equality. Also at the beginning of 20th century the life of female scientists was a very hard one:

Is mathematics politically correct?

- The following anecdote about Emmy Noether (1882 – 1935), (up to now) the most important female mathematician in the whole history of mathematics is taken from the same biography of Hilbert from which I cited a story about a mathematician-turned-novelist. The following was about what happened when the mathematicians around Hilbert tried to provide Noether the habilitaion(*) in 1915:

> Now he [=Hilbert] answered their [=University Senate's] formal argument against habilitating Emmy Noether with equal directedness: "**Meine Herren I do not see that the sex of the candidate is an argument against her permission as a Privatdozent. After all, the Senate is not a bathhouse.**"
>
> — Constance Reid, "Hilbert", (1969).

(*) Habilitation is a kind of second doctorate which is the main requirement to obtain a professorship in German university system — a researcher with habilitation but not yet a professorship is called a "Privatdozent".

Is mathematics politically correct?

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Emmy Noether (1882 – 1935) was the founder of the modern algebra (which is something totally different from school algebra!). With her preservation theorem, she is also well-known among theoretical physicists.

Is mathematics politically correct?

- Mathematics today is more international and there are more female mathematics professors.
- Mathematicians around me as an example:
  - My thesis adviser was a German female professor.
  - Among my co-authors of mathematical papers there are 2 female mathematicians.
  - The nationality and number of my co-authors of mathematical papers and monographs are:

Brazil (?)	Germany (4)	Hungary (3)	Israel (3)
Japan (5)	Mexico (1)	Poland (2)	Russia (1)
Slovakia (1)	U.S.A. (1)		

Recommended links and books

- MacTutor History of Mathematics archive of School of Mathematics and Statistics University of St Andrews, Scotland is written in simple English and is very much to recommend. Some of the pictures used in the slides above are taken from there.
- If you can read Japanese, my Japanese translation of ,,Was sind und was sollen die Zahlen" by Richard Dedekind (1831–1916) and commentary may give you some insight into the the foundation of mathematics also from the point of view of today (complemented by my commentary).
- If you know high school mathematics and a little bit of university level mathematics (such as the beginnings of calculus and linear algebra) and if you are interested in mathematical proofs, Proofs from THE BOOK is very much to recommend. A caution: this book is quite self contained but also very difficult. If you have worked out many chapters of this book, you can compete well with the best graduates of math departments.
- A Mathematician's Apology (1940) by G.H. Hardy.

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Topics dropped due to the lack of time:

- Abstraction in Mathematics
- Comupter science and mathematics
- Compouter and mathematics

Thank you for your attention.