Reflection numbers under large continuum

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Reflection numbers

- For a class $\mathcal{C}$ of structures with a notion $\sqsubseteq_{\mathcal{C}}$ of substructures, $A \in \mathcal{C}$ and a cardinal $\kappa \leq |\,A\,|$, let

$$S^{\mathcal{C}}_{<\kappa}(A) = \{B \in \mathcal{C} \,:\, B \sqsubseteq_{\mathcal{C}} A,\ |\,B\,| < \kappa\}.$$

  We identify elements of $\mathcal{C}$ with their underlying sets and consider $S^{\mathcal{C}}_{<\kappa}(A) \subseteq [A]^{<\kappa}$.

- ▷ We assume that $S^{\mathcal{C}}_{<\kappa}(A)$ contains a club $\subseteq [A]^{<\kappa}$ for all $A \in \mathcal{C}$.

- For $\mathcal{C}$ as above and a property P, the **reflection number of P in $\mathcal{C}$** is defined as:

$$\mathfrak{Refl}(\mathcal{C}, P) = \begin{cases} \min\{\kappa \in \mathrm{Card} \,:\, \text{for all } A \in \mathcal{C}, \text{ if } A \not\models P \text{ then} \\ \qquad \text{there are club many } B \in S^{\mathcal{C}}_{<\kappa}(A) \\ \qquad \text{such that } B \not\models P\}, \qquad \text{if } \{\kappa \in \mathrm{Card} \,:\, \cdots\} \neq \emptyset; \\ \infty, \qquad \text{otherwise.} \end{cases}$$

Reflection numbers (2/2)

- **If the property P is hereditary**, i.e. if $A \sqsubseteq_{\mathcal{C}} B$ and $B \models P$ always impies $A \models P$ then the reflection number can be more simply represented as:

$$\mathfrak{Refl}(\mathcal{C}, P) = \begin{cases} \min\{\kappa \in \mathrm{Card} : \text{for all } A \in \mathcal{C} \text{ if } A \not\models P \text{ then} \\ \qquad \text{there is } B \in S^{\mathcal{C}}_{<\kappa}(A) \\ \qquad \text{such that } B \not\models P\}, \\ \qquad\qquad \text{if } \{\kappa \in \mathrm{Card} : \cdots\} \neq \emptyset\,; \\ \infty, \qquad\qquad \text{otherwise.} \end{cases}$$

Examples of reflection numbers

$\mathcal{C} =$ trees;
$P \Leftrightarrow$ special

- The assertion $\mathfrak{Refl}(\mathcal{C}, P) = \aleph_2$ is known as **Rado's Conjecture**.
- We shall denote this reflection number with $\mathfrak{Refl}_{\text{Rado}}$.
- $\aleph_1 < \mathfrak{Refl}_{\text{Rado}} \leq \infty$. $V = L$ ($\square_\kappa$ for class many κ) $\Rightarrow \mathfrak{Refl}_{\text{Rado}} = \infty$.
- $\mathfrak{Refl}_{\text{Rado}} = \aleph_2$ can be forced starting from a strongly compact cardinal (Todorcěvić 1983).
- $\mathfrak{Refl}_{\text{Rado}} = \aleph_2$ implies strong forms of Chang's Conjecture (Todorčević 1993, Doebler 2013, S.F.-Sakai-Torres-Usuba ∞).
- $\mathfrak{Refl}_{\text{Rado}} = \aleph_2$ implies $2^{\aleph_0} \leq \aleph_2$ (Todorčević 1993).
- $\mathfrak{Refl}_{\text{Rado}} = \aleph_2$ implies the Fodor-type Reflection Principle (FRP) and hence all consequences of FRP like SCH (S.F.-Rinot 2011), stationarity reflection (of sets of ordinals of countable cofinality) etc. (S.F.-Sakai-Torres-Usuba ∞).

Examples of reflection numbers (2/3)

$\mathcal{C} =$ partial orderings;
$P \Leftrightarrow$ union of countably many chains (w.r.t. the partial ordering)

- The assertion $\mathfrak{Refl}(\mathcal{C}, P) = \aleph_2$ for these $\mathcal{C}$ and P is known as **Galvin's Conjecture**.
- ▷ It is still **open** if Galvin's Conjecture is consistent.
- ▷ We shall denote this reflection number with $\mathfrak{Refl}_{\text{Galvin}}$.

$\mathcal{C} =$ graphs;
$P \Leftrightarrow$ of countable chromatic number

- We shall denote the reflection number $\mathfrak{Refl}(\mathcal{C}, P)$ for these $\mathcal{C}$ and P with $\mathfrak{Refl}_{\text{chr}}$
- $\beth_\omega < \mathfrak{Refl}_{\text{chr}}$ (Erdős and Hajnal 1966).
- We have

$$\mathfrak{Refl}_{\text{Rado}} \leq \mathfrak{Refl}_{\text{Galvin}} \leq \mathfrak{Refl}_{\text{chr}} \leq \omega_1\text{-strongly compact cardinal.}$$

Examples of reflection numbers (3/3)

$\mathfrak{Refl}_{\text{Rado}} \leq \mathfrak{Refl}_{\text{Galvin}} \leq \mathfrak{Refl}_{\text{chr}} \leq \omega_1$-strongly compact cardinal.

- κ is called the **ω_1-strongly compact cardinal** if it is the smallest cardinal κ with the property that for any $\mathcal{L}_{\omega_1,\omega}$ theory T, whenever all subtheories of T of size $< \kappa$ are satisfiable ($< \kappa$-satisfiable) then T itself is also satisfiable.

$\mathcal{C} =$ Boolean algebras;
$P \Leftrightarrow$ free

- We denote the reflection number $\mathfrak{Refl}(\mathcal{C}, P)$ for these $\mathcal{C}$ and P by $\mathfrak{Refl}^{Ba}_{\text{free}}$. Similarly $\mathfrak{Refl}^{gp}_{\text{free}}$ and $\mathfrak{Refl}^{agp}_{\text{free}}$ for groups and abelian groups.

▷ $\aleph_1 < \mathfrak{Refl}^{Ba}_{\text{free}}, \mathfrak{Refl}^{gp}_{\text{free}}, \mathfrak{Refl}^{agp}_{\text{free}} \leq \infty$

▷ **Open.** Can $\mathfrak{Refl}^{Ba}_{\text{free}}, \mathfrak{Refl}^{gp}_{\text{free}}, \mathfrak{Refl}^{agp}_{\text{free}}$ be different?

▷ $\mathfrak{Refl}^{gp}_{\text{free}}, \mathfrak{Refl}^{agp}_{\text{free}} \leq \omega_1$-strongly compact cardinal.

▷ **Open?** $\mathfrak{Refl}^{Ba}_{\text{free}} \leq \omega_1$-strongly compact cardinal ?

Large Continuum

- $2^{\aleph_0}$ can be consistently "very large" in the following sense:

> There is an inner model M with the same cardinals s.t. $2^{\aleph_0}$ is a large cardinal in M.
>
> There is a generic elementary embedding with the critical point $= 2^{\aleph_0}$.

- This can be forced by starting from a model V with a vary large cardinal (e.g. a supercompact cardinal) κ and then adding κ many reals in a "coherent" way.

- The existence of a certain generic elementary embedding together with the indestructiblility of the negation of the property involved implies the inequality $\mathfrak{Refl} \leq 2^{\aleph_0}$!

Yet another reflection number

$\mathcal{C} =$ first countable topological spaces;
$P =$ metrizable

- The question about the consistency of $\mathfrak{Refl}(\mathcal{C}, P) = \aleph_2$ is known as **Hamburger's Problem**. This is also a well-known longstanding open question. We shall call $\mathfrak{Refl}(\mathcal{C}, P)$ for these $\mathcal{C}$ and P the reflection number of Hamburger's Problem and denote is as $\mathfrak{Refl}_{\mathrm{HP}}$.

- $\aleph_1 < \mathfrak{Refl}_{\mathrm{HP}} \leq \infty$. $\mathfrak{Refl}_{\mathrm{HP}} = \infty$ is consistent.

- $\mathfrak{Refl}_{\mathrm{HP}} \leq$ the ω_1-strongly compact cardinal (if it exits).

Theorem 0 (Dow-Tall-Weiss 1990). $\mathfrak{Refl}_{\mathrm{HP}} \leq 2^{\aleph_0}$ is consistent with very large continuum. More precisely this holds if strongly compact many Cohen reals are added.

Remark. In the construction for the theorem above both $\mathfrak{Refl}_{\mathrm{HP}} < 2^{\aleph_0}$ and $\mathfrak{Refl}_{\mathrm{HP}} = 2^{\aleph_0}$ are possible.

Sketch of a consistency proof of Rado's Conjecture

- Suppose that κ is strongly compact and $\mathbb{P} = \mathrm{Col}(\kappa, \omega_1)$. We show that $\Vdash_{\mathbb{P}}$ "$\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_2$".
- Let G be $(\mathbb{P}, V)$-generic and $T \in V[G]$ a tree s.t. (*) $V[G] \models \forall T' \in [T]^{<\aleph_2}$ is special. Note that $(\aleph_2)^{V[G]} = \kappa$.
- ▷ We have to show: $V[G] \models T$ is special.
- In $V[G]$, let $\lambda = |\,T\,|$. Let $j : V \overset{\prec}{\to} M$ be the strongly compact embedding with $j(\kappa) > \lambda$. Let $\mathbb{P}^* = j(\mathbb{P})$ and let G^* be a $(\mathbb{P}^*, V)$-generic set with $G \subseteq G^*$.
- ▷ Let $j^* : V[G] \overset{\prec}{\to} M[G^*]$; $[\underset{\sim}{a}]^G \mapsto [j(\underset{\sim}{a})]^{G^*}$. Let $T^* = j^*(T)$ and let T' be s.t. $j^{*\,\prime\prime}T \subseteq T'$ and $T' \in [T^*]^{\aleph_1} \cap M[G^*]$. Thus $M[G^*] \models T' \in [T^*]^{<\aleph_2}$. By elementarity of j^* and (*), $M[G^*] \models T'$ is special. Hence $V[G^*] \models j''T \cong T$ is special.
- By the following Lemma, T is special even in $V[G]$:

> **Lemma 1** (Todorčević 1983). For any tree T and σ-closed p.o. $\mathbb{Q}$ if $\Vdash_{\mathbb{Q}}$ "T is special" then T is special.

Cohen forcing

Lemma 1 (Todorčević 1983). For any tree T and σ-closed p.o. $\mathbb{Q}$ if $\Vdash_{\mathbb{Q}}$ " T is special " then T is special.

Proposition 2. For any tree T and $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$ for any κ if $\Vdash_{\mathbb{P}}$ " T is special " then T is special.

Proof.

- If $\kappa \leq 2^{\aleph_0}$ then $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$ is σ-centered and $\Vdash_{\mathbb{P}}$ " T is special " clearly implies that T is special.
- For $\kappa > 2^{\aleph_0}$, suppose that $\Vdash_{\mathbb{P}}$ " T is special ". Let $\mathbb{Q} = \mathrm{Col}(\kappa^+, \omega_1)$.

  We have $\Vdash_{\mathbb{P} * \underset{\sim}{Q}}$ " T is special " where $\underset{\sim}{Q}$ is s.t. $\mathbb{Q} * \mathbb{P} \cong \mathbb{P} * \underset{\sim}{Q}$.

  Since $\Vdash_{\mathbb{Q}}$ " $\mathbb{P}$ is σ-centered " it follows that $\Vdash_{\mathbb{Q}}$ " T is special ".

  Thus, by Todorcevic's Lemma 1, T is special. □

Adding strongly compact many Cohen reals

Proposition 2. For any tree T and $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$ for any κ if $\Vdash_{\mathbb{P}}$ " T is special " then T is special.

- Similarly to the consistency proof of Rado's conjecture, Proposition 2 above can be used to prove:

Theorem 3. If κ is a strongly compact cardinal then, letting $\mathbb{P} = \mathrm{Fn}(\lambda, 2)$ for some $\lambda \geq \kappa$, we have $\Vdash_{\mathbb{P}}$ " $\mathfrak{Refl}_{\mathsf{Rado}} = \kappa$ ". In particular, assertions

"$\mathfrak{Refl}_{\mathsf{Rado}} = 2^{\aleph_0}$ + the continuum is very large" and

"$\mathfrak{Refl}_{\mathsf{Rado}} < 2^{\aleph_0}$ + the continuum is very large"

are both consistent.

- Remember $\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_2 \Rightarrow 2^{\aleph_0} \leq \aleph_2$ (Todorcevic).

Question. (M. Viale) $\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_3 \Rightarrow 2^{\aleph_0} \leq \aleph_3$?

Indesctructible reflection numbers

- For a class $\mathcal{C}$ of structures and a property P, let us say that $A \in \mathcal{C}$ is **indectructibly** $\neg P$ if $\Vdash_{\mathbb{P}}$ " $A \models \neg P$ " for any σ-closed p.o. $\mathbb{P}$.
- For a class $\mathcal{C}$ of structures and a property P, the **indestructible reflection number of** P **in** $\mathcal{C}$ is defined as:

$$\mathfrak{Refl}^*(\mathcal{C}, P) = \begin{cases} \min\{\kappa \in \mathrm{Card} : \text{if } A \in \mathcal{C} \text{ is indestructively } \neg P \text{ then there are club many } B \in S^{\mathcal{C}}_{<\kappa}(A) \text{ such that } B \not\models P\}, & \text{if } \{\kappa \in \mathrm{Card} : \cdots\} \neq \emptyset; \\ \infty, & \text{otherwise.} \end{cases}$$

- Let $\mathfrak{Refl}^*_{\text{Galvin}}$ and $\mathfrak{Refl}^*_{\text{chr}}$ be $\mathfrak{Refl}^*$ variations of $\mathfrak{Refl}_{\text{Galvin}}$ and $\mathfrak{Refl}_{\text{chr}}$.

$$\begin{array}{ccccc} & & \mathfrak{Refl}_{\text{Galvin}} & \leq & \mathfrak{Refl}_{\text{chr}} \\ & & \vee\!\mid & & \vee\!\mid \\ \mathfrak{Refl}_{\text{Rado}} & \leq & \mathfrak{Refl}^*_{\text{Galvin}} & \leq & \mathfrak{Refl}^*_{\text{chr}}. \end{array}$$

▷

Further consistency results

- $$\begin{array}{ccccc} & & \mathfrak{Refl}_{\mathsf{Galvin}} & \leq & \mathfrak{Refl}_{\mathsf{chr}} \\ & & \text{\rotatebox{90}{$\geq$}} & & \text{\rotatebox{90}{$\geq$}} \\ \mathfrak{Refl}_{\mathsf{Rado}} & \leq & \mathfrak{Refl}^*_{\mathsf{Galvin}} & \leq & \mathfrak{Refl}^*_{\mathsf{chr}}. \end{array}$$

- Arguments similar to that of Theorem 3. amounts to the following theorems:

Theorem 4. For a strongly compact cardinal κ and $\mathbb{P} = \mathrm{Col}(\omega_1, \kappa)$, we have $\Vdash_{\mathbb{P}}$ "$\mathfrak{Refl}^*_{\mathsf{chr}} = \aleph_2$".

Theorem 5. For a strongly compact cardinal κ and $\mathbb{P} = \mathrm{Fn}(\lambda, 2)$ for $\lambda \geq \kappa$, we have $\Vdash_{\mathbb{P}}$ "$\mathfrak{Refl}^*_{\mathsf{chr}} \leq \kappa \leq \lambda = 2^{\aleph_0}$".

Theorem 6. For a measurable cardinal κ and $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$, we have
$\Vdash_{\mathbb{P}}$ "for any graph Γ of size continuum and uncountable chromatic number there exists a subgraph of size $<$ continuum with uncountable chromatic number".

Martin's Axiom

- Let $\mathfrak{ma}$ be the first number of dense sets for which Martin's Axiom fails. Thus $\omega_1 \leq \mathfrak{ma} \leq 2^{\aleph_0}$.

Theorem 7. (Baumgartner-Malitz-Reihhardt, 1970) Any tree of size $< \mathfrak{ma}$ without uncountable chain is special. □

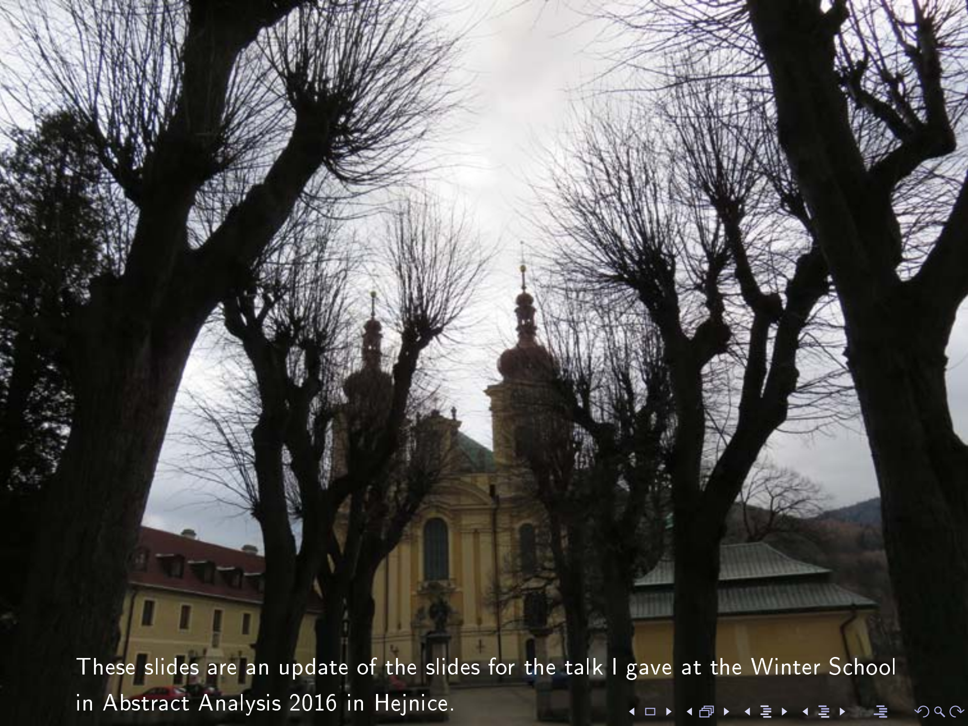
- Let $T_{\mathbb{R}} = \{t : t$ is a strictly increasing sequences in $\mathbb{R}$ of successor length $< \omega_1\}$ be considered as a tree with endextension.

Theorem 8. (Todorcevic, 1983) $T_{\mathbb{R}}$ is not special.

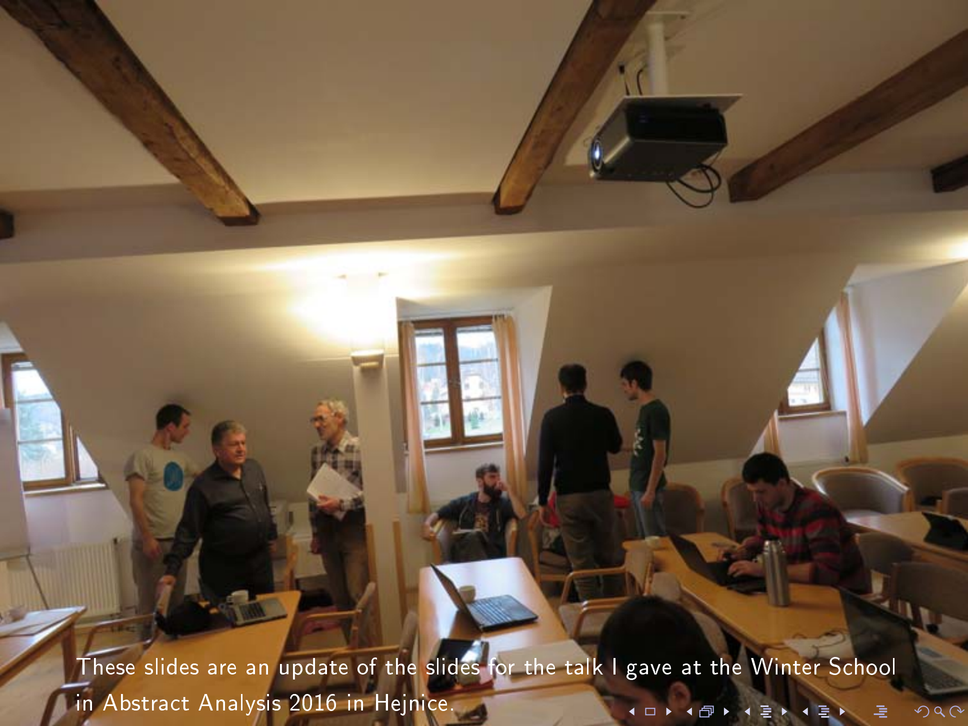
Corollary 9. $\mathfrak{ma} < \mathfrak{Refl}_{\mathsf{Rado}}$.

Theorem 10. (Folklore, (S.F., 1992)) If $\mathbb{P}$ has the c.c.c. then, for any A in an universal algebra $\mathcal{C}$, if $\Vdash_{\mathbb{P}}$ "A is free" then A is free.

Corollary 11. If κ is a supercompact cardinal then for the canonical c.c.c. p.o. $\mathbb{P}$ forcing $\kappa = 2^{\aleph_0}$ and $\mathfrak{ma} = 2^{\aleph_0}$ (i.e. MA), $\Vdash_{\mathbb{P}}$ "$\mathfrak{Refl}^{\mathcal{C}}_{\mathsf{free}} \leq 2^{\aleph_0}$" for any universal algebra $\mathcal{C}$. In particular $\mathfrak{Refl}^{\mathcal{C}}_{\mathsf{free}} < \mathfrak{Refl}_{\mathsf{Rado}}$ is consistent.



These slides are an update of the slides for the talk I gave at the Winter School in Abstract Analysis 2016 in Hejnice.



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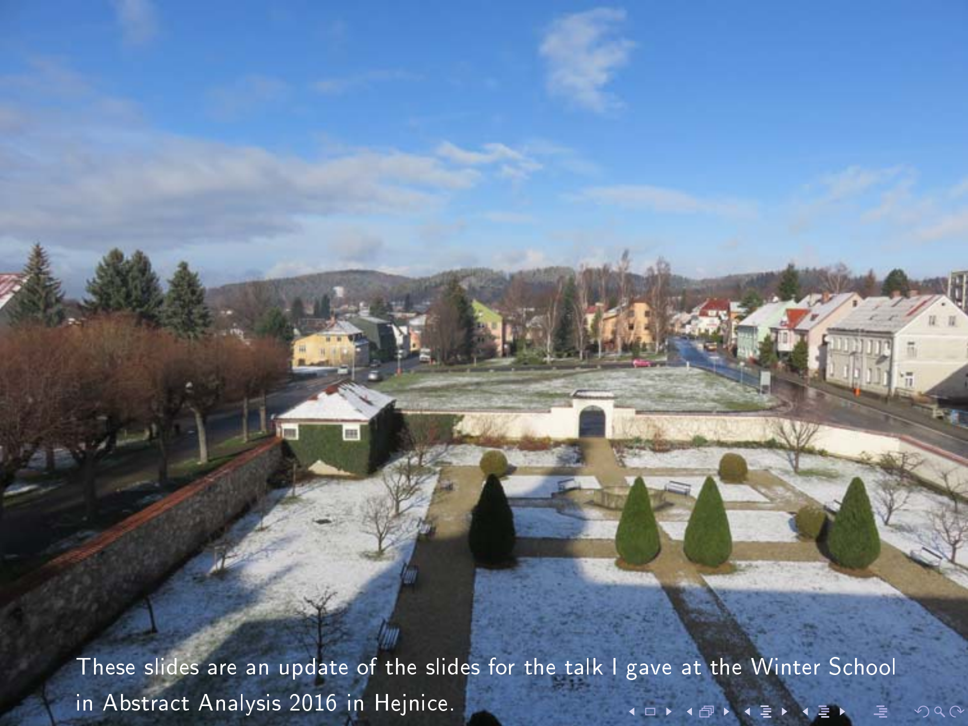

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$2^{\aleph_0}$

Děkuji vám za pozornost !