Fodor-type Reflection Principle and very weak square principles

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This talk is based on ...

a joint work (in progress) with Hiroshi Sakai (酒井 拓史) against the background of the results obtained in the recent joint researches with

- I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba (2010).
- H. Sakai, L. Soukup and T. Usuba (preprint).
- A. Rinot (2011? to appear).

FRP

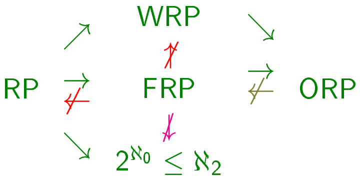
- **Fodor-type Reflection Principle (FRP)** is the principle asserting that the following FRP(κ) holds for all regular cardinal $\kappa \geq \aleph_2$:

> FRP(κ): *For any stationary* $S \subseteq E^{\kappa}_{\omega}$ *and* $g : S \to [\kappa]^{\aleph_0}$ *there is* $I \in [\kappa]^{\aleph_1}$ *such that*
>
> - $\mathrm{cf}(I) = \omega_1$;
> - $g(\alpha) \subseteq I$ *for all* $\alpha \in I \cap S$;
> - *for any* $f : S \cap I \to \kappa$ *s.t.* $f(\alpha) \in g(\alpha) \cap \alpha$ *for all* $\alpha \in S \cap I$, *there is* $\xi^* < \kappa$ *s.t.* $f^{-1}{}''\{\xi^*\}$ *is stationary in* $sup(I)$.

- $E^{\kappa}_{\lambda} = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$.

- MA$^+$(σ-closed) $\to$ Axiom R $\to$ RP $\nearrow$ WRP $\searrow$ ORP; RP $\searrow$ FRP $\nearrow$ ORP

Separation of FRP from other reflection principles



- F., I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba*
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$*$ FRP is preserved by ccc foricng

Separation of FRP from other reflection principles at $\aleph_2$

$$
\begin{array}{ccccccc}
 & \swarrow\!\!\!\!\!\!\!/ & & \mathrm{WRP}([\aleph_2]^{\aleph_0}) & & \nwarrow\!\!\!\!\!\!\!/ & \\
 & \nearrow & & & & \searrow & \\
\mathrm{RP}([\aleph_2]^{\aleph_0}) & & & \not\uparrow \quad \not\downarrow & & & \mathrm{ORP}(\aleph_2) \\
 & \searrow & & & & \nearrow & \\
 & \nwarrow\!\!\!\!\!\!\!/ & & \mathrm{FRP}(\aleph_2) & & \swarrow\!\!\!\!\!\!\!/ &
\end{array}
$$

- J. Krueger
  T. Usuba
- T. Miyamoto*
- T. Usuba

$*$ $\mathit{consis}(\mathrm{ZFC} + \mathrm{FRP}(\aleph_2)) \quad \leftrightarrow \quad \mathit{consis}(\mathrm{ZFC} + \exists\text{mahlo cardinal})$

Characterizations of FRP

- FRP is equivalent to the most of the known "mathematical" reflection theorems. In particular, FRP is equivalent to each of the following assertions:

  ▷ For any locally countably compact topological space X, if all subspaces Y of X cardinality $\leq \aleph_1$ are metrizable then X itself is metrizable. (F., Sakai, Soukup and Usuba (preprint); Z. Balogh (2002) under Axiom R)

  ▷ For a locally separable, countably tight space X, if all subspaces Y of X of cardinality $\leq \aleph_1$ are meta-Lindelöf then X itself is meta-Lindelöf. (F., Sakai, Soukup and Usuba (preprint))

  ▷ For a T_1 space with point countable base, if all subspaces Y of X of cardinality $\leq \aleph_1$ are left-separated then X itself is left-separated. (F.; W. Fleissner (1986) under Axiom R)

  ▷ For a countably tight space of local density $\aleph_1$, if all subspaces Y of X of cardinality $\leq \aleph_1$ are collectionwise Hausdorff then X itself is collectionwise Hausdorff.
(F., Sakai, Soukup and Usuba (preprint); Fleissner (1986) under Axiom R)

Characterizations of FRP

- FRP is also equivalent to the following assertions:

  ▷ For any graph $G = (G, \varepsilon)$, if every subgraphs of G of cardinality $\leq \aleph_1$ have countable coloring number, then the coloring number of G itself is also countable.
  (F., Sakai, Soukup and Usuba (preprint); Fleissner (1986) under Axiom R)

  ▷ For any Boolean algebra A, if there are club many openly generated subalgebras B of A of cardinality $\leq \aleph_1$, then A is openly generated. (F. and Rinot (2011?); F. (1994) under Axiom R)

Other consequences of FRP

- FRP implies $\neg\square_\kappa$ for all cardinals $\kappa \geq \aleph_1$.

- FRP implies $\neg\mathsf{ADS}_\kappa$ for all singular cardinals κ of countable cofinality ω. Hence:

  - FRP implies $\neg\square^*_\kappa$ for all singular cardinals κ of countable cofinality.

  - FRP implies SSH .

(Very) weak squares

Let κ be a singular cardinal.

$\square^*_\kappa$**:** (Weak square, Jensen) There is a sequence $\langle \mathcal{C}_\alpha : \alpha \in Lim(\kappa^+)\rangle$ of non empty sets s.t. for every $\alpha \in Lim(\kappa^+)$

- ▷ $\mathcal{C}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{C}_\alpha| \leq \kappa$;
- ▷ every $C \in \mathcal{C}_\alpha$ is club in α and if $\mathrm{cf}(\alpha) < \kappa$ then $otp(C) < \kappa$;
- ▷ for every $C \in \mathcal{C}_\alpha$ and $\delta \in (C)'$, we have $C \cap \delta \in \mathcal{C}_\delta$.

VWS$_\kappa$: (Very weak square, Foreman and Magidor) There is a sequence $\langle C_\alpha : \alpha < \kappa^+\rangle$ and a club $D \subseteq \kappa^+$ s.t. for every $\alpha \in D$

- ▷ $C_\alpha \subseteq \alpha$, C_α is unbounded in α;
- ▷ for all bounded $x \in [C_\alpha]^{<\omega_1}$, there is $\beta < \alpha$ s.t. $x = C_\beta$.

▶ E.g. under GCH, VWS$_\kappa$ follows from $\square^*_\kappa$ (Foreman and Magidor).

(Very) weak squares (2/2)

VWS$_\kappa$: (Very weak square, Foreman and Magidor) There is a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ and a club $D \subseteq \kappa^+$ s.t. for every $\alpha \in D$

- ▷ $C_\alpha \subseteq \alpha$, C_α is unbounded in α;
- ▷ for all bounded $x \in [C_\alpha]^{<\omega_1}$, there is $\beta < \alpha$ s.t. $x = C_\beta$.

$\square^{***}_{\omega_1,\kappa}$: (F. and Soukup) There is a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ and a club set $D \subseteq \kappa^+$ s.t. for $\alpha \in D$ with $\mathrm{cf}(\alpha) \geq \omega_1$

- ▷ $C_\alpha \subseteq \alpha$, C_α is unbounded in α;
- ▷ $[\alpha]^{<\omega_1} \cap \{C_{\alpha'} : \alpha' < \alpha\}$ dominates $[C_\alpha]^{<\omega_1}$ (w.r.t. $\subseteq$).

Some applications of $\Box^{***}_{\aleph_1,\lambda}$

Theorem 1 (F. and Soukup, 1997)

Suppose that SSH and $\Box^{***}_{\omega_1,\kappa}$ holds for all uncountable cardinals κ of countable cofinality. Then any Boolean algebra B has the weak Freese-Nation property if, for any/some sufficiently large regular θ and for any internally unbounded $M \prec \mathcal{H}(\theta)$ with $A \in M$, $A \cap M$ is a σ-complete subalgebra of A.

Theorem 2 (F. and Soukup, 1997)

Assume GCH and $\Box^{***}_{\omega_1,\kappa}$ holds for all uncountable cardinals κ of countable cofinality. then for any cardinal λ, and $\mathbb{P} = \mathrm{Fn}(\lambda, 2)$, $\Vdash_{\mathbb{P}}$ "$\mathcal{P}(\omega)$ has the weak Freese-Nation property".

Theorem 3 (F. and Soukup, 1997; F., Geschke, Shelah and Soukup, 2001)

The assertions of both of the theorems above are refuted under $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

FRP and $\square^{***}_{\omega_1,\kappa}$

Proposition 4

If "ZFC + ∃ a supercompact cardinal is consistent then so is "ZFC + $\mathrm{MA}^+(\sigma$-closed) + VWS_κ for all singular κ"

Proof.
- We can extend the ground model with a supercompact κ to a model where VWS_κ for all singular κ holds and κ is still supercompact (Foreman and Magidor, 1997).
- By collapsing κ to $\aleph_2$ by $Col(\aleph_1, < \kappa)$ we obtain a model of ZFC + $\mathrm{MA}^+(\sigma$-closed) + VWS_κ. □ (**Proposition 4**)

Corollary 5

"FRP + $\square^{***}_{\omega_1,\kappa}$ for all singular cardinal" is consistent (relative to a supercompact). In particular, e.g. "FRP + MA + ¬CH + $\square^{***}_{\omega_1,\kappa}$ for all singular cardinal" is consistent.

Proof. Note that "FRP + $\square^{***}_{\omega_1,\kappa}$ for all singular cardinal" is preserved under c.c.c. forcing. □ (**Corollary 5**)

FRP and $\square^{***}_{\omega_1,\kappa}$

▶ Under "FRP + $\square^{***}_{\omega_1,\kappa}$ for all singular cardinal", both of the following hold:

▷ For any Boolean algebra A, if there are club many openly generated subalgebras B of A of cardinality $\leq \aleph_1$, then A is openly generated.

▷ Any Boolean algebra B has the weak Freese-Nation property if, for any/some sufficiently large regular θ and for any internally unbounded $M \prec \mathcal{H}(\theta)$ with $A \in M$, $A \cap M$ is a σ-complete subalgebra of A.

FRP and $\square(\lambda)$

Theorem 6 (H. Sakai)

Under GCH + MA^{+}(σ-closed), for any regular $\lambda \geq \aleph_2$, there is a poset forcing "FRP(λ) and $\square(\lambda)$".

Theorem 7 (B. Velicikovic (+ Sakai))

WRP *implies* $\neg\square(\lambda)$ *for all regular* $\lambda \geq \aleph_2$.

Vielen Dank für Ihre Aufmerksamkeit!

This file and the version for the presentation as well as some of the preprints mentioned in the talk are downloadable from:

http://kurt.scitec.kobe-u.ac.jp/~fuchino/

Reflection principles

- **MA$^+$(σ-closed):** for every σ-closed poset $\mathbb{P}$, set $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| = \aleph_1$ and a $\mathbb{P}$-name $\dot{S}$ of a stationary subset of ω_1, there exists a $\mathcal{D}$-generic filter G over $\mathbb{P}$ s.t. $\dot{S}^G = \{\alpha : p \Vdash_{\mathbb{P}} \text{“}\alpha \in \dot{S}\text{”} \text{ for some } p \in G\}$ is a stationary subset of ω_1.

- **Axiom R:** For all cardinals $\kappa > \aleph_1$, for any stationary $S \subseteq [\kappa]^{\aleph_0}$ and an ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, there is $I \in \mathcal{T}$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

  ▷ $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$ is said to be **ω_1-club** if it is unbounded w.r.t. $\subseteq$ and closed w.r.t. union of $\subseteq$-chain of length ω_1.

  ▷ Axiom R corresponds to the reflection to internally unbounded.

Reflection principles (2/2)

- **Axiom R :** For all cardinals $\kappa > \aleph_1$, for any stationary $S \subseteq [\kappa]^{\aleph_0}$ and an ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, There is $I \in \mathcal{T}$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

- **RP :** For all cardinals κ of uncountable cofinality and for any stationary $S \subseteq [\kappa]^{\aleph_0}$, there is $I \in [\kappa]^{\aleph_1}$ s.t. $\omega_1 \subseteq I$, $\mathrm{cf}(I) = \omega_1$ and $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

- **WRP :** For all cardinals $> \aleph_1$, and for any stationary $S \subseteq [\kappa]^{\aleph_0}$, there is $I \in [\kappa]^{\aleph_1}$ s.t. $\omega_1 \subseteq I$, and $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

- **ORP :** For all cardinals $> \aleph_1$, and for any stationary $S \subseteq E^{\kappa}_{\omega}$, there is $I \in [\kappa]^{\aleph_1}$ s.t. $\omega_1 \subseteq I$, $\mathrm{cf}(I) = \omega_1$ and $S \cap I$ is stationary in $\sup I$.

openly generated Ba's

- For a Boolean algebra B and a subalgebra A of B, A is a **relatively complete subalgebra** of B if $\sup_A(A \restriction b)$ exists for all $b \in B$.
- A Boolean algebra A is **openly generated** if $A \cap M$ is relatively complete in A for any $M \prec \mathcal{H}(\theta)$ with $A \in M$ for sufficiently large regular θ.

- For a Boolean algebra B and a subalgebra A of B, A is a **σ-complete subalgebra** of B if if $A \restriction b$ is a countably generated ideal of A for all $b \in B$.
- A Boolean algebra B has the **weak Freese-Nation property** if $A \cap M$ is σ-complete in A for any uncountable $M \prec \mathcal{H}(\theta)$ with $A \in M$ for sufficiently large regular θ.

ADS_κ and $\mathsf{ADS}^-(\lambda)$

ADS_κ: there is a sequence $\langle a_\alpha : \alpha < \kappa^+\rangle$ s.t.

- $a_\alpha \subseteq \kappa$, $\sup(a_\alpha) = \kappa$ and $otp(a_\alpha) = \mathrm{cf}(\kappa)$ for all $\alpha < \kappa^+$;
- for any $\beta < \kappa^+$, there is a mapping $f : \beta \to \kappa$ s.t. $a_\alpha \setminus f(\alpha)$, $\alpha < \beta$ are pairwise disjoint.

$\mathsf{ADS}^-(\lambda)$: there are a stationary set $S \subseteq \lambda$ and a sequence $\langle a_\alpha : \alpha \in S\rangle$ s.t.

- $a_\alpha \subseteq \alpha$ and $otp(a_\alpha) = \omega$ for all $\alpha \in S$;
- for any $\beta < \lambda$, there is a mapping $f : S \cap \beta \to \beta$ s.t. $f(\alpha) < \sup(a_\alpha)$ for all $\alpha \in S \cap \beta$ and $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ are pairwise disjoint.

- If $\mathrm{cf}(\kappa) = \omega$. Then ADS_κ implies $\mathsf{ADS}^-(\kappa^+)$.
- $\mathsf{FRP}(<\lambda)$ (i.e. $\mathsf{FRP}(\kappa)$ for all regular $\kappa \in \lambda \setminus \aleph_2$) $\Leftrightarrow \neg\mathsf{ADS}^-(\kappa)$ holds for all regular $\kappa < \lambda$.

(F., Juhász, Soukup, Szentmiklóssy and Usuba, 2010)

- Shelah's Strong Hypothesis (**SSH**) is the assertion equivalent to the following:

  ▷ For every uncountable cardinal κ of countable cofinality, we have $cf([\kappa]^{\aleph_0}, \subseteq) = \kappa^+$.

- By the characterization of SSH above , Singular Cardinal Hypothesis follows from SSH.