Topological Characterization of Shelah's Strong Hypothesis and Fodor-type Reflection Principle

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東北大学 ロジック セミナー での講演

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Axiomatist reading of "Reverse Mathematics"

A "reverse" reading of Reverse Mathematics' philosophy in a broad sense from axiomatist point of view:

> A (combinatorial) principle may be considered as prominent if it is equivalent to many "natural" "mathematical" statements over a base theory.

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Axiom of Choice (AC) is equivalent to each of the following statements over ZF:

- Well-ordering theorem
- Zorn's lemma
- Existence of a basis to each vector space
- Tychonoff's theorem etc.

The assertion (axiom) "there exists a weakly compact cardinal" is equivalent to each of the following statements over ZFC:

- There exists an inaccessible cardinal with tree property.
- There exists an inaccessible cardinal κ s.t. $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem.
- There exists a Π^1_1-indescribable cardinal. etc.

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There can be also the following variation of the "reverse" reading of Reverse Mathematics' philosophy in a broader sense from axiomatist point of view:

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> A (combinatorial) principle may be considered as prominent if it is equiconsistent to many "natural" "mathematical" statements over a base theory.

AC is equi-consistent with the following statement over ZF:

- $1 + 1 = 2$.

The assertion (axiom) "there exists a weakly compact cardinal" is equi-consistent with the following statements over ZFC:

- There is no ω_2-Aronszajn tree
  (J.H. Silver, W.J. Mitchell 1972/73)
- Every stationary set $S \subseteq E^{\omega_2}_{\omega_0}$ reflects at almost all $E^{\omega_2}_{\omega_1}$
  (M. Magidor 1982),

Remark. In the last two statements above, there is no mention at all on "large" cardinals!!

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- For regular κ and $\lambda > \kappa$, $E^{\lambda}_{\kappa} = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa\}$.
- A stationary set $S \subseteq \lambda$, S reflects at $\alpha < \lambda$ if $S \cap \alpha$ is stationary in α.
- "stationary set $S \subseteq E^{\omega_2}_{\omega_0}$ reflects at almost all $E^{\omega_2}_{\omega_1}$" means here that there is a closed unbounded $C \subseteq \lambda$ s.t.

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Fodor-type Reflection Principle (FRP) is the principle which asserts that the following $\mathrm{FRP}(\kappa)$ holds for all regular uncountable κ:

FRP(κ): For any stationary $S \subseteq E^{\kappa}_{\omega} = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ and $g : S \to [\kappa]^{\leq\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

- $\mathrm{cf}(I) = \omega_1$;
- $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- for any regressive $f : S \cap I \to \kappa$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

Remark. By the last "▶", S reflects at $\sup(I)$.

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Fodor-type Reflection Principle (2/3)

- FRP follows from RP.

(F., Juhász, Soukup, Szentmiklóssy and Usuba, 2010)

RP: For any cardinal λ of cofinality $> \omega_1$ and stationary $S \subseteq [\lambda]^{\aleph_0}$, there is an $I \in [\lambda]^{\aleph_1}$ s.t.

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- $[X]^\kappa = \{x \subseteq X : |x| = \kappa\}$.
- $C \subseteq [X]^\kappa$ is closed unbounded if C is cofinal in $[X]^\kappa$ w.r.t. $\subseteq$ and closed w.r.t. union of $\subseteq$-chain of length $\leq \kappa$.
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Known implications among related combinatorial principles:

$$\text{Martin's Maximum} \Rightarrow \text{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \text{RP} \Rightarrow \text{FRP}$$

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The consistency of this principle follows from Con(ZFC+ there exists a supercompact cardinal)

The last implication is irreversible !!!

- RP implies $2^{\aleph_0} \leq \aleph_2$ while FRP is compatible with arbitrary (consistent) size of the continuum.

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(F., Juhász, Soukup, Szentmiklóssy and Usuba, 2010)

Fodor-type Reflection Principle (3/3)

Known implications among related combinatorial principles:

$$\text{Martin's Maximum} \Rightarrow \text{MA}^+(\sigma\text{-closed}) \Rightarrow \text{Axiom R} \Rightarrow \text{RP} \Rightarrow \text{FRP}$$

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Assertions equivalent to FRP (1/3)

Theorem 1 (F., Juhász, Soukup, Szentmiklóssy and Usuba, (2010); F., Sakai, Soukup and Usuba, preprint (201?))

FRP *is equivalent to the following assertion over* ZFC*:*

For a locally countably compact topological space X, if X is $\leq \aleph_1$-metrizable then X is metrizable.

- A topological space X is countably compact if every countable open cover of X has a finite subcover.
- A topological space X is locally countably compact if every point has a closed neighborhood which is countably compact.
- X is $\leq \kappa$-metrizable for a cardinal κ if every subspace Y of X of size $\leq \kappa$ is metrizable.
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FRP *is equivalent to the following assertion over* ZFC*:*

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Theorem 1 asserts that FRP is equivalent to the assertion: "For a locally countably compact topological space X, if X is $\leq\aleph_1$-metrizable then X is metrizable".

▶ If we drop "locally" from the assertion above, we obtain a theorem in ZFC:

Theorem 4 (Alan Dow, 1988)

For a countably compact topological space X, if X is $\leq\aleph_1$-metrizable then X is metrizable. □

▷ "$\leq\aleph_1$" cannot be replaced by "$\leq\aleph_0$":

ω_1 (the first uncountable ordinal ($=\aleph_1$ as a set)) with the canonical order topology is countably compact, first countable, and $\leq\aleph_0$-metrizable but not metrizable.

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The order topology of ω_1 (1/3)

Claim 4.1

ω_1 with the canonical order topology is countably compact.

Proof. Suppose that O_k, $k \in \omega$ are open subsets of ω_1 s.t.

(1) $\omega_1 = \bigcup_{k\in\omega} O_k$.

We show first that $\omega_1 \setminus O_k$ is bounded for some $k \in \omega$. Suppose otherwise. Then $\omega_1 \setminus O_k$, $k \in \omega$ are all closed and unbounded. It follows that $\omega_1 \setminus \bigcup_{k\in\omega} O_k = \bigcap_{k\in\omega}(\omega_1 \setminus O_k)$ is also closed and unbounded; hence non empty in particular. This is a contradiction to (1).

We may assume that $\omega_1 \setminus O_0$ is bounded. Assume now, toward a contradiction, that $\bigcup_{k<i} O_k \neq \omega_1$ for all $i \in \omega$. For $i \in \omega$ let $\alpha_i < \omega_1$ be s.t.

$$\alpha_i \in \omega_1 \setminus \bigcup_{k\leq i} O_k \quad \text{but} \quad (\alpha_i, \omega_1) \subseteq \bigcup_{k\leq i} O_k.$$

Then $\langle \alpha_i : i \in \omega \rangle$ is decreasing and it is strictly decreasing at infinitely many places. A contradiction. □

The order topology of ω_1 (2/3)

Claim 4.2

ω_1 with the canonical order topology is first countable.

Proof. For $\alpha \in \omega_1$, if α is a successor ordinal then α is an isolated point. Otherwise α has the countable neighborhood base:

$$\{(\beta, \alpha + 1) \, : \, \beta < \alpha\}.$$

□

Claim 4.3

ω_1 with the canonical order topology is $\leq \aleph_0$-metrizable.

Proof. For any countable $Y \subseteq \omega_1$, there is $\alpha < \omega_1$ s.t. $Y \subseteq \alpha$. But since α (with its canonical order) is an order preserving embedding of α into $\mathbb{R}$, α is metrizable and hence also Y. □

The order topology of ω_1 (3/3)

Claim 4.4

ω_1 with the canonical order topology is not metrizable.

Proof. Suppose that there is a metric d which induces the order topology of ω_1.
For all $\alpha \in Lim(\omega_1)$, let $n_\alpha \in \omega \setminus \{0\}$ be s.t.
$B_d(\alpha, \frac{2}{n_\alpha}) \subseteq \alpha + 1 = (-1, \alpha + 1)$ and $\beta_\alpha < \alpha$ be s.t.
$\beta_\alpha \in B_d(\alpha, \frac{1}{n_\alpha})$.
By Fodor's lemma there is $n^* \in \omega$ and $\beta^* < \omega_1$ s.t.

$$S = \{\alpha \in Lim(\omega_1) : n_\alpha = n^* \text{ and } \beta_\alpha = \beta^*\}$$

is stationary and hence, in particular, infinite.
Let $\alpha_0, \alpha_1 \in S$ be s.t. $\alpha_0 < \alpha_1$. Then we have

$$d(\alpha_0, \alpha_1) \leq d(\alpha_0, \beta^*) + d(\beta^*, \alpha_1) \leq \frac{1}{n_{\alpha_0}} + \frac{1}{n_{\alpha_1}} = \frac{2}{n^*}.$$

Thus, $\alpha_1 \in B_d(\alpha_0, \frac{2}{n^*}) = B_d(\alpha_0, \frac{2}{n_{\alpha_0}}) \subseteq \alpha_0 + 1$. This is a contradiction. □

Background of Theorem 1 (2)

The statement of Theorem 1 is a natural generalization of Dow's Theorem:

> For a locally countably compact topological space X,
> if X is $\leq \aleph_1$-metrizable then X is metrizable.

It has been known that this statement is independent form ZFC:

- If we assume $V = L$ (the axiom asserting that the set-theoretic universe consists of constructible sets in the sense of Gödel) then the assertion above is false.
- Zoltan Balogh (posth. 2002) showed that Axiom R (recall that principle is e.g. a consequence of Martin's Maximum) implies the assertion above (Balogh's metrization theorem).

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Shelah's Strong Hypothesis

- The proof of the equivalence of FRP with the assertion on openly generated Boolean algebras used the fact that FRP implies Shelah's Strong Hypothesis (SSH)

- Shelah's Strong Hypothesis (SSH) is the principle equivalent to the assertion: For every cardinal κ we have $\mathrm{cf}([\kappa^+]^{\aleph_0}, \subseteq) = \kappa^+$ where
  - κ^+ denotes the successor cardinal of κ.
  - $\mathrm{cf}(A, \leq)$ for a partial ordering $\langle A, \leq \rangle$ is the smallest cardinality of $B \subseteq A$ cofinal in A (i.e., $\forall x \in A \exists y \in B (x \leq y)$).

- "Shelah's Strong Hypothesis" is actually not so strong! It is merely slightly stronger than "Singular Cardinal Hypothesis" (SCH).

Theorem 5 (F. and Rinot, submitted (201?))

FRP *implies* SSH. *In particular,* FRP *implies* SCH. □

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SSH also enjoys a topological characterization

- We call a topological space X thin if for every $D \subseteq X$ we have $|\overline{D}| \leq |D|^+$. X is $<\kappa$-thin for a cardinal κ if $|\overline{D}| \leq |D|^+$ holds for all $D \subseteq X$ of cardinality $< \kappa$.
- A topological space X is countably tight if for every $Y \subseteq X$ and $x \in X$ if $x \in \overline{Y}$ then there is a countable $Y' \subseteq Y$ s.t. $x \in \overline{Y'}$.

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SSH *is equivalent with the following assertion:*

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The assertion (axiom) "there exists a weakly compact cardinal" is equi-consitstent with the following statement over ZFC:

- Every stationary set $S \subseteq E^{\omega_2}_{\omega_0}$ reflects at almost all $E^{\omega_2}_{\omega_1}$ (M. Magidor 1982),

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Thank you for your attention!

My preprints and papers mentioned in the talk are available at:

http://kurt.scitec.kobe-u.ac.jp/~fuchino/preprints.html

This slide will be linked to:

http://kurt.scitec.kobe-u.ac.jp/~fuchino/