

Topological Characterization of Shelah's Strong Hypothesis and Fodor-type Reflection Principle

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(November 27, 2010 (15:03 JST) version)

東北大学 ロジック セミナー での講演

July 23, 2010. 15:30 ~ 17:[12][0-9]

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Axiomatist reading of "Reverse Mathematics" Top. char. of SSH and FRP (2/21)

A "reverse" reading of Reverse Mathematics' philosophy in a broad sense from axiomatist point of view:

A (combinatorial) principle may be considered as prominent if it is equivalent to many "natural" "mathematical" statements over a base theory.

Axiom of Choice (AC) is equivalent to each of the following statements over ZF:

- ▶ Well-ordering theorem
- ▶ Zorn's lemma
- ▶ Existence of a basis to each vector space
- ▶ Tychonoff's theorem etc.

The assertion (axiom) “there exists a weakly compact cardinal” is equivalent to each of the following statements over ZFC:

- ▶ There exists an inaccessible cardinal with tree property.
- ▶ There exists an inaccessible cardinal κ s.t. $\mathcal{L}_{\kappa, \omega}$ satisfies the Weak Compactness Theorem.
- ▶ There exists a Π_1^1 -indescribable cardinal. etc.

There can be also the following **variation** of the "reverse" reading of Reverse Mathematics' philosophy in a broader sense from axiomatist point of view:

A (combinatorial) principle may be considered as prominent if it is equiconsistent to many "natural" "mathematical" statements over a base theory.

AC is equi-consistent with the following statement over ZF:

- ▶ $1 + 1 = 2$.

The assertion (axiom) “there exists a weakly compact cardinal” is equi-consistent with the following statements over ZFC:

- ▶ There is no ω_2 -Aronszajn tree
(J.H. Silver, W.J. Mitchell 1972/73)
- ▶ Every stationary set $S \subseteq E_{\omega_0}^{\omega_2}$ reflects at almost all $E_{\omega_1}^{\omega_2}$
(M. Magidor 1982),

Remark. In the last two statements above, there is no mention at all on “large” cardinals!!

The assertion (axiom) “there exists a weakly compact cardinal” is equi-consistent with the following statement over ZFC:

- ▶ Every stationary set $S \subseteq E_{\omega_0}^{\omega_2}$ reflects at almost all $E_{\omega_1}^{\omega_2}$ (M. Magidor 1982),

- ▶ For regular κ and $\lambda > \kappa$, $E_\kappa^\lambda = \{\alpha < \lambda : \text{cf}(\alpha) = \kappa\}$.
- ▶ A stationary set $S \subseteq \lambda$, S reflects at $\alpha < \lambda$ if $S \cap \alpha$ is stationary in α .
- ▶ “stationary set $S \subseteq E_{\omega_0}^{\omega_2}$ reflects at almost all $E_{\omega_1}^{\omega_2}$ ” means here that there is a closed unbounded $C \subseteq \lambda$ s.t.

$$\{\alpha \in E_{\omega_1}^{\omega_2} : S \text{ reflects at } \alpha\} \supseteq C \cap E_{\omega_1}^{\omega_2}.$$

Fodor-type Reflection Principle (FRP) is the principle which asserts that the following $\text{FRP}(\kappa)$ holds for all regular uncountable κ :

FRP(κ): For any stationary $S \subseteq E_\omega^\kappa = \{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$ and $g : S \rightarrow [\kappa]^{\leq \aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

- ▶ $\text{cf}(I) = \omega_1$;
- ▶ $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- ▶ for any regressive $f : S \cap I \rightarrow \kappa$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}''\{\xi^*\}$ is stationary in $\text{sup}(I)$.

Remark. By the last "▶", S reflects at $\text{sup}(I)$.

▶ FRP follows from RP.

(F., Juhász, Soukup, Szentmiklóssy and Usuba, 2010)

RP: For any cardinal λ of cofinality $> \omega_1$ and stationary $S \subseteq [\lambda]^{\aleph_0}$, there is an $I \in [\lambda]^{\aleph_1}$ s.t.

▶ $\omega_1 \subseteq I$;

▶ $\text{cf}(I) = \omega_1$;

▶ $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

▶ $[X]^\kappa = \{x \subseteq X : |x| = \kappa\}$.

▶ $C \subseteq [X]^\kappa$ is **closed unbounded** if C is cofinal in $[X]^\kappa$ w.r.t. \subseteq and closed w.r.t. union of \subseteq -chain of length $\leq \kappa$.

▶ $S \subseteq [X]^\kappa$ is **stationary** if $S \cap C \neq \emptyset$ holds for any closed unbounded $C \subseteq [X]^\kappa$.

Known implications among related combinatorial principles:

Martin's Maximum \Rightarrow $MA^+(\sigma\text{-closed}) \Rightarrow$ Axiom R \Rightarrow RP \Rightarrow FRP

↑

*The consistency of this principle follows from
Con(ZFC+ there exists a supercompact cardinal)*

The last implication is irreversible !!!

▶ RP implies $2^{\aleph_0} \leq \aleph_2$ while FRP is compatible with arbitrary (consistent) size of the continuum.

(F., Juhász, Soukup, Szentmiklóssy and Usuba, 2010)

Theorem 1 (F., Juhász, Soukup, Szentmiklóssy and Usuba, (2010); F., Sakai, Soukup and Usuba, preprint (201?))

FRP is equivalent to the following assertion over ZFC:

For a locally countably compact topological space X ,
if X is $\leq \aleph_1$ -metrizable then X is metrizable.

- ▶ A topological space X is **countably compact** if every countable open cover of X has a finite subcover.
- ▶ A topological space X is **locally countably compact** if every point has a closed neighborhood which is countably compact.
- ▶ X is **$\leq \kappa$ -metrizable** for a cardinal κ if every subspace Y of X of size $\leq \kappa$ is metrizable.
- ▶ There are at least 4 other statements in topology known to be equivalent to FRP.

► For an infinite graph $G = \langle G, \mathcal{E} \rangle$, the coloring number of G ($col(G)$) is defined as

$$col(G) = \min\{\mu : \\ \text{there is a well-ordering } \prec \text{ of } G \text{ s.t.} \\ |\{y \in G : y \prec x \text{ and } \{x, y\} \in \mathcal{E}\}| < \mu \text{ for all } x \in G\}.$$

Theorem 2 (F., Sakai, Soukup and Usuba, preprint (201?))

FRP is equivalent to the following assertion over ZFC:

For any infinite graph $G = \langle G, \mathcal{E} \rangle$, if
every subgraph H of G of cardinality $\leq \aleph_1$ satisfies
 $col(H) \leq \aleph_0$
then $col(G) \leq \aleph_0$.

► A Boolean algebra B is **openly generated** if there is a mapping $f : B \rightarrow [B]^{<\aleph_0}$ s.t., for any $b, c \in B$ with $b \leq c$, there is $d \in f(b) \cap f(c)$ s.t. $b \leq d \leq c$.

Theorem 3 (F. and Rinot, submitted (201?))

FRP is equivalent to the following assertion over ZFC:

*For any Boolean algebra B ,
if there are closed-unboundedly many openly generated subalgebras C of B of cardinality $\leq \aleph_1$ then B is openly generated.*

► The proof of this theorem uses the fact that FRP implies Shelah's Strong Hypothesis (SSH)

Theorem 1 asserts that FRP is equivalent to the assertion: “For a locally countably compact topological space X , if X is $\leq \aleph_1$ -metrizable then X is metrizable”.

► If we drop “locally” from the assertion above, we obtain a theorem in ZFC:

Theorem 4 (Alan Dow, 1988)

For a countably compact topological space X , if X is $\leq \aleph_1$ -metrizable then X is metrizable.



▷ “ $\leq \aleph_1$ ” cannot be replaced by “ $\leq \aleph_0$ ”:

ω_1 (the first uncountable ordinal ($= \aleph_1$ as a set)) with the canonical order topology is countably compact, first countable, and $\leq \aleph_0$ -metrizable but not metrizable.

Claim 4.1

ω_1 with the canonical order topology is countably compact.

Proof. Suppose that O_k , $k \in \omega$ are open subsets of ω_1 s.t.

$$(1) \omega_1 = \bigcup_{k \in \omega} O_k.$$

We show first that $\omega_1 \setminus O_k$ is bounded for some $k \in \omega$.

Suppose otherwise. Then $\omega_1 \setminus O_k$, $k \in \omega$ are all closed and unbounded. It follows that $\omega_1 \setminus \bigcup_{k \in \omega} O_k = \bigcap_{k \in \omega} (\omega_1 \setminus O_k)$ is also closed and unbounded; hence non empty in particular. This is a contradiction to (1).

We may assume that $\omega_1 \setminus O_0$ is bounded. Assume now, toward a contradiction, that $\bigcup_{k < i} O_k \neq \omega_1$ for all $i \in \omega$. For $i \in \omega$ let $\alpha_i < \omega_1$ be s.t.

$$\alpha_i \in \omega_1 \setminus \bigcup_{k \leq i} O_k \text{ but } (\alpha_i, \omega_1) \subseteq \bigcup_{k \leq i} O_k.$$

Then $\langle \alpha_i : i \in \omega \rangle$ is decreasing and it is strictly decreasing at infinitely many places. A contradiction. □

Claim 4.2

ω_1 with the canonical order topology is first countable.

Proof. For $\alpha \in \omega_1$, if α is a successor ordinal then α is an isolated point. Otherwise α has the countable neighborhood base:

$$\{(\beta, \alpha + 1) : \beta < \alpha\}.$$

□

Claim 4.3

ω_1 with the canonical order topology is $\leq \aleph_0$ -metrizable.

Proof. For any countable $Y \subseteq \omega_1$, there is $\alpha < \omega_1$ s.t. $Y \subseteq \alpha$. But since α (with its canonical order) is an order preserving embedding of α into \mathbb{R} , α is metrizable and hence also Y .

□

Claim 4.4

ω_1 with the canonical order topology is not metrizable.

Proof. Suppose that there is a metric d which induces the order topology of ω_1 .

For all $\alpha \in \text{Lim}(\omega_1)$, let $n_\alpha \in \omega \setminus \{0\}$ be s.t.

$B_d(\alpha, \frac{2}{n_\alpha}) \subseteq \alpha + 1 = (-1, \alpha + 1)$ and $\beta_\alpha < \alpha$ be s.t.

$\beta_\alpha \in B_d(\alpha, \frac{1}{n_\alpha})$.

By Fodor's lemma there is $n^* \in \omega$ and $\beta^* < \omega_1$ s.t.

$$S = \{\alpha \in \text{Lim}(\omega_1) : n_\alpha = n^* \text{ and } \beta_\alpha = \beta^*\}$$

is stationary and hence, in particular, infinite.

Let $\alpha_0, \alpha_1 \in S$ be s.t. $\alpha_0 < \alpha_1$. Then we have

$$d(\alpha_0, \alpha_1) \leq d(\alpha_0, \beta^*) + d(\beta^*, \alpha_1) \leq \frac{1}{n_{\alpha_0}} + \frac{1}{n_{\alpha_1}} = \frac{2}{n^*}.$$

Thus, $\alpha_1 \in B_d(\alpha_0, \frac{2}{n^*}) = B_d(\alpha_0, \frac{2}{n_{\alpha_0}}) \subseteq \alpha_0 + 1$. This is a contradiction. □

Background of Theorem 1 (2)

The statement of Theorem 1 is a natural generalization of Dow's Theorem:

For a locally countably compact topological space X ,
if X is $\leq \aleph_1$ -metrizable then X is metrizable.

It has been known that this statement is independent form ZFC:

- ▶ If we assume $V = L$ (the axiom asserting that the set-theoretic universe consists of constructible sets in the sense of Gödel) then the assertion above is false.
- ▶ Zoltan Balogh (posth. 2002) showed that **Axiom R** (recall that principle is e.g. a consequence of **Martin's Maximum**) implies the assertion above (**Balogh's metrization theorem**).

▶ The proof of the equivalence of FRP with the assertion on openly generated Boolean algebras used the fact that **FRP implies Shelah's Strong Hypothesis (SSH)**

▶ **Shelah's Strong Hypothesis (SSH)** is the principle equivalent to the assertion: For every cardinal κ we have $\text{cf}([\kappa^+]^{\aleph_0}, \subseteq) = \kappa^+$ where

▷ κ^+ denotes the successor cardinal of κ .

▷ $\text{cf}(A, \leq)$ for a partial ordering $\langle A, \leq \rangle$ is the smallest cardinality of $B \subseteq A$ cofinal in A (i.e., $\forall x \in A \exists y \in B (x \leq y)$).

▶ “Shelah's Strong Hypothesis” is actually not so strong! It is merely slightly stronger than “**Singular Cardinal Hypothesis (SCH)**”.

Theorem 5 (F. and Rinot, submitted (201?))

FRP implies SSH. In particular, FRP implies SCH. □

- ▶ We call a topological space X **thin** if for every $D \subseteq X$ we have $|\overline{D}| \leq |D|^+$. X is **$<\kappa$ -thin** for a cardinal κ if $|\overline{D}| \leq |D|^+$ holds for all $D \subseteq X$ of cardinality $< \kappa$.
- ▶ A topological space X is **countably tight** if for every $Y \subseteq X$ and $x \in X$ if $x \in \overline{Y}$ then there is a countable $Y' \subseteq Y$ s.t. $x \in \overline{Y'}$.

Theorem 6 (F. and Rinot, submitted (201?))

SSH is equivalent with the following assertion:

*For any countably tight topological space X
if X is $<\aleph_1$ -thin then X is thin.*

The assertion (axiom) “*there exists a weakly compact cardinal*” is equi-consistent with the following statement over ZFC:

- ▶ Every stationary set $S \subseteq E_{\omega_0}^{\omega_2}$ reflects at almost all $E_{\omega_1}^{\omega_2}$ (M. Magidor 1982),

Theorem 7 (Miyamoto, (2010))

The assertion (axiom) “there exists a Mahlo cardinal” is equi-consistent with $\text{FRP}(\aleph_2)$ over ZFC:

Thank you for your attention!

My preprints and papers mentioned in the talk are available at:

<http://kurt.scitec.kobe-u.ac.jp/~fuchino/preprints.html>

This slide will be linked to:

<http://kurt.scitec.kobe-u.ac.jp/~fuchino/>