Gödel's Speed-up Theorem and its impacts on Mathematics

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http://fuchino.ddo.jp/slides/speed-up-theorem-2016-pf.pdf

A rough statement of the theorem

For a concretely given (recursive) theory T with certain property, in particular s.t. the elementary arithmetic can be developed in T, and any computable (recursive) function $f : \mathbb{N} \to \mathbb{N}$, there is a formula $\varphi = \varphi(x)$ in the language of the theory T s.t. for each $n \in \mathbb{N}$, $\varphi(\underline{n})$ is provable from T but the simplest proof of $\varphi(\underline{n})$ has the degree (of complication) $\geq f(n)$. In contrast, $T + consis(T)$ proves $\forall x \varphi(x)$ and thus there is a linear function g s.t. the degree of the proof $\varphi(\underline{n})$ from $T + consis(T)$ is $\leq g(n)$.

- $\underline{n}$ denotes the numeral (in the language of T) representing n.
- The assertion above varies according to the exact choice of sertain property and degree (of complication).

History of the theorem

- Kurt Gödel (1906–1978, (明治 39 年–昭和 53 年)) mentioned the statement of his Speed-up Theorem in a seminar report in 1936 (昭和 11 年).
- The proof of Gödel's Incompleteness Theorems were obtained in 1930. The Speed-up Theorem can be seen as a spin-off of the results around the Incompleteness Theorems.
- ▷ Both of the terms "incompleteness theorem" and "speed-up theorem" were coined not by Gödel himself but introduced soon after these results were public.
- Gödel never published the proof of the Speed-up Theorem.
- Samuel Buss' paper in 1995 contains one of the first explicit proof of the Gödel's theorem.

History of the theorem (2/2)

- The original statement of the theorem was as follows:

> Sei nun S_i das System der Logik i-ter Stufe, wobei die natürlichen Zahlen als Individuen betrachtet werden. ... Zu jeder in S_i berechenbaren Funktion ϕ gibt es unendlich viele Formeln f von der Art, daß, wenn k die Länge eines kürzesten Beweises für f in S_i und ℓ die Länge eines kürzesten Beweises für f in S_{i+1} ist, $k > \phi(\ell)$.
>
> K. Gödel [1936]

> Now let S_i be the system of the ith order logic where the natural numbers are considered to be the basic objects. ... To each computable function ϕ in S_i, there are infinitely many formulas f s.t., if k is the length of a shortest proof of f in S_i and ℓ the length of a shortest proof of f in S_{i+1}, then we have $k > \phi(\ell)$.
>
> translated by S.F.

Another version of the Speed-up Theorem

- The version of the Speed-up Theorem with

  degree = the length of the proof (= number of the formulas involved in the proof),

  as in the original formulation of the theorem by Gödel, is dependent on the system of the proof.

  - It can be even false in some artificially set deduction system!

- The version of the theorem with

  degree = the sum of the lengths of the formulas appearing in the proof

  is independent of the choice of the deduction system (as far as the language of the theory contains only finitely many non logical sysmbols):

Another version of the Speed-up Theorem (2/2)

- Let $\mathcal{L}_{\{\}}$ be the language consisting of $\in$, $\{.,.\}$, $\emptyset$. Let $\mathsf{ZF}_{\{\}}$ be the axiom system of Zermelo-Fraenkel set theory formulated in $\mathcal{L}_{\{\}}$.

Theorem 1 Let T be a concretely given (recursive) theory containing a large enough fragment of the theory $\mathsf{ZF}_{\{\}}$. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a computable (recursive) function. Then there is an $\mathcal{L}_{\{\}}$-formula $\varphi(x_1)$ s.t., for each $n \in \mathbb{N}$, we have $T \vdash \varphi(\underline{n})$ but, if $T \vdash^{P} \varphi(\underline{n})$ for a proof P in T, then $T \vdash \mathit{rank}(\ulcorner P \urcorner) \geq f(\underline{n})$.

In contrast we have
$T + \mathit{consis}(\ulcorner\ulcorner T \urcorner\urcorner) \vdash \forall n \in \omega\ \varphi(n)$.

Mathematical and philosophical consequences of the Speed-up Theorem

- Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a fast growing computable function s.t., say, $f(7)$ exceeds the number of atoms in the whole universe.
  - Let T be as in Theorem 1 and $\varphi = \varphi(x)$ be as in Theorem 1 for these f and T. Then we know that $T \vdash \varphi(\underline{7})$ but it is impossible to write down the proof (as far as T is consistent).
  - In $T + consis(\ulcorner\ulcorner T\urcorner\urcorner)$ we obtain a proof of $\varphi(\underline{7})$ of reasonable length!
- Let T and φ be as above (and assume that T is consistent).
  - The theory $\tilde{T} = T + \neg\varphi(\underline{7})$ is inconsistent but there is no feasible proof of the inconsistency!

Large cardinals and infinitely many times speed-up

- There are (recursive) theories T_i, $i < \omega_1^{CK}$ s.t. $T_0 = \mathrm{ZFC}$, $\langle Th(T_i) : i < \omega_1^{CK} \rangle$ is continuously increasing $T_{i+1} \vdash consis(\ulcorner\ulcorner T_i \urcorner\urcorner)$ for all $i < \omega_1^{CK}$ and $Th(\bigcup_{i<\omega_1^{CK}} T_i) \subseteq Th(\mathrm{ZFC} +$ "there is an incaccessible cardinal"$)$
- Similar assertion holds between two extensions of set theory T, T' where the stronger theory T' include a large cardinal axiom which transcends the weaker set theory T.
- Compare this with Gödel's Speed-up Theorem.

Incomplete reference

- 渕野 昌，集合論 (= 数学) の未解決問題，in:

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