集合論的多元宇宙と様相論理

Set-theoretic multiverse and modal logic

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The printer version of these slides is going to be downloadable as
http://kurt.scitec.kobe-u.ac.jp/~fuchino/slides/summer-school2015-multiverse-pf.pdf

Foundation of the multiverse of set generic extensions

Where can we consider the multiverse of set generic extensions of a fixed ground model?

That is, **what** is the outer universe (meta-universe) which accommodate all the universes of set theory accessible from the ground model in finite steps of set forcing extension and the converse (set forcing ground).

The 1. possible setting: Seen from the "outer universe" $\models \mathrm{ZFC}$, "the ground model" is a countable transitive model of ZFC.

A problematic of the 1. setting:

The assertion "there is a model of ZFC" is not provable in ZFC. Even if we assume "there is a model of ZFC" there is no guarantee that there is a countable transitive model of ZFC!

See:

http://kurt.scitec.kobe-u.ac.jp/~fuchino/notes/woodin-incompl-e.pdf

Foundation of the multiverse of set generic extensions

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Possible solutions:

1.1. We replace ZFC with a sufficiently large finite subset of ZFC. (For a finite subset T of ZFC, it is a theorem in ZFC that there is a countable transitive model of $\ulcorner T \urcorner$)

— This does not give a foundation of multiverse theory since "sufficiently large" depends on the poset with which the generic extension is constructed!

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Possible solutions:

1.2. We introduce a new constant symbol m and add the axioms "m is countable and transitive" and "$m \models \ulcorner\varphi\urcorner$" for each axiom φ of ZFC to the axiom system ZFC (equiconsistent with ZFC by compactness)

— a countable transitive model is too much skolem-paradoxical to be considered as THE ground model in which we "live"!.

Foundation of the multiverse of set generic extensions (2/4)

The 2. possible setting: We remain in the ground model and consider the class of posets, complete embeddings between them and forcing relation on them (and nothing else) as a (class) Kripke frame in which modality is interpreted.

- Mathematically this setting is quite satisfactory.

  — However, the resulting theory in this framework is purely algebraic and the multiverse becomes merely a mode of parlance!

Foundation of the multiverse of set generic extensions (3/4)

The 3. possible setting: We fix a Grothendieck universe V_κ (where κ is strongly inaccessible) and think that our ground model is V_κ while our outer universe is $V[G]$ where G is the $(V, \mathbb{P})$-generic filter for the Lévy collapse $\mathbb{P} = \mathrm{Col}(\omega, \kappa)$:

- $$\mathrm{Col}(\omega,\kappa) = \{p : p \text{ is a function } \wedge\ |p| < \aleph_0\ \wedge\ \mathrm{dom}(p) \subseteq \kappa \times \omega \wedge \forall \langle \alpha, n\rangle \in \mathrm{dom}(p)\ (p(\alpha, n) = 0 \vee p(\alpha, n) \in \alpha)\}$$
  ordered by $p \leq q \ \Leftrightarrow\ p \supseteq q$.
- For a generic G set over $\mathrm{Col}(\omega, \kappa)$, $g = \bigcup G$ is a mapping $g : \kappa \times \omega \to \kappa$ s.t., for each $\alpha < \kappa$, $g(\cdot, \alpha) : \omega \to \alpha$ and it is a surjection. Hence, all $\alpha < \kappa$ are countable in $V[G]$.
- By regularity of κ, κ remains a cardinal in $V[G]$. Thus, we have $V[G] \models \kappa = \omega_1$.
- For all poset $\mathbb{Q}$ in V_κ, $|\mathbb{Q}| = \aleph_0$ in $V[G]$ and there are only countably many dense subsets of $\mathbb{Q} \in V_\kappa$ (seen in $V[G]$). Thus there is a $(\mathbb{P}, V)$-generic set H in $V[G]$ and κ is still strongly inaccessible in $V[H]$.

Foundation of the multiverse of set generic extensions (4/4)

The 4. possible setting (a modification of the 3. setting):

The outer universe is a model of ZC (or ZC + a weak form of Replacement) while the ground model is an inner model of the outer universe.

- Let κ, $\mathbb{P}$ and G as in the previous slide. Our outer universe is $(V_{\omega_1})^{V[G]}$ while the ground model is V_κ. We formulate an appropriate axiom system (with new unary predicate for elements of the ground model — in von Neumann-Bernays-Gödel style) and work in it. Von Neumann-Bernays-Gödel style of formulation is needed so that we can talk about the class (of classes) of definable inner models.

Definability of the ground model of a set generic extension

Theorem (R. Laver, 2007) There is a fixed formula $\Phi(x, y)$ in $\mathcal{L}_{\mathrm{ZF}}$ s.t., if $V = W[G]$ for an inner model W and generic G over W then $W = \{x : \Phi(x, a)\}$ for some a.

Use and misuse of modal logic in multiverse theory

- The lectures "強制法と様相論理" at 数学基礎論サマースクール 2015 was actually an introduction to the theory of forcing disguised as a tutorial on the relationship between modal logic and set-theoretic multiverse.
- Modal logic in connection with set-theoretic multiverse may help in finding new set-theoretic axioms (statements) and enable discussions on the meaning and the significance of the new axioms.

  Example: Stavi-Väänänen principle.
- There are many natural variations of multiverses (e.g. universes obtained by c.c.c. forcing, proper forcing etc. The real (not necessarily set generic) multiverses, ...) What about multi-modality corresponding to these variations?
- What about the axiomatic treatment of multiverses (e.g. axioms not necessarily formalizable in ZFC setting but in some second order set theory)?

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