反復強制法入門

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講義の目標と方針

- 反復強制法 (**iterated forcing**) の最も簡単な場合である

  有限サポート反復 (**finite support iteration**, **FS-iteration**)

  を導入する.

- FS-iteration を用いて Martin's Axiom ($+ \neg$CH) の (相対的) 無矛盾性証明をする.

- (おまけ) Martin's Axiom の応用をいくつか見てみる.

- Slides では (ネット上に置いてあるプリンター版でも) "…", "…" は clickable texts である.

- 以下, slides と板書は英語, 口頭ではできるだけ日本語で行なう.

General Setting

- We use the setting in which our ground model is V. When we are talking about $V[G]$ and $V[G] \models \varphi(\dot{a}^G, \dots)$ in this setting, we are actually talking about $V^{\mathbb{P}}$ and $p \Vdash_{\mathbb{P}}$ "$\varphi(\dot{a}, \dots)$".

- Nevertheless, we sometimes switch to the setting with a countable transitive model M of (some finite fragment of) ZFC as the ground model, and talk about the (real) generic extension $M[G]$ of M to obtain better "semantic" understanding of what is happing with the corresponding p.o. $\mathbb{P}$ and $M^{\mathbb{P}}$.

  Even in the latter setting, we simply continue to call our ground model V and pretend as if this would be the real set theoretic universe: E.g., we will simply talk about $\mathcal{P}(\kappa)$ meaning $\mathcal{P}(\kappa)^V$.

- Thus, when we are talking about the V in this sense, we sometimes also think that the generic sets over V are taken from the "hyperuniverse" W outside our universe V s.t. "$V \in W$" and W thinks that V is countable and transitive — although we never need to talk about such W.

P.o.s

- $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}}, 1\!\!1_{\mathbb{P}} \rangle$ is a **p.o.** if $\leq_{\mathbb{P}}$ is a preordering (i.e. transitive and reflexive) with a maximal element $1\!\!1_{\mathbb{P}}$.
- $\mathbb{P}$-names (i.e. elements of $V^{\mathbb{P}}$) are denoted with dotted alphabets like $\dot{a}$, $\dot{b}$,... $\dot{A}$, $\dot{B}$,... etc. $\dot{G}$ denotes the (generic) name $\{\langle \check{p}, p \rangle : p \in \mathbb{P}\}$ of a $(V, \mathbb{P})$-generic set.
- p, $q \in \mathbb{P}$ are **incompatible** if there is no $r \in \mathbb{P}$ s.t. $r \leq_{\mathbb{P}} p, q$. We write $p \perp_{\mathbb{P}} q$ to denote this. If p and q are not incompatible then they are **compatible** (notation: $p \not\perp_{\mathbb{P}} q$).
- $A \subseteq \mathbb{P}$ is an **antichain** in $\mathbb{P}$ if elements of A are pairwise incompatible. A is a **maximal antichain** in $\mathbb{P}$ if there is no antichain in $\mathbb{P}$ containing A as a proper subset. For $X \subseteq \mathbb{P}$ a **maximal antichain** $A \subseteq X$ is an antichain maximal (w.r.t. $\subseteq$) among antichains $\subseteq X$.
- ▷ If $A \subseteq \mathbb{P}$ is a maximal antichain, $D_A = \{q \in \mathbb{P} : q \leq_{\mathbb{P}} p, \in A\}$ is a dense open subset of $\mathbb{P}$. On the other hand, if D is a dense subset of $\mathbb{P}$ a maximal antichain $A \subseteq D$ is a maximal antichain in $\mathbb{P}$. If D is also open, we have $D_A \subseteq D$.

Complete embeddings of p.o.s

- For p.o.s $\mathbb{P}$ and $\mathbb{Q}$, a mapping $i : \mathbb{P} \to \mathbb{Q}$ is said to be a **complete embedding** if (a) i is 1-1, (b) $i(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{Q}}$, (c) i is order preserving and incompatibility preserving, and
  (d) for any maximal antichain $A \subseteq \mathbb{P}$, $i''A$ is always a maximal antichain in $\mathbb{Q}$.

  (For a mapping f on X and $S \subseteq X$, $f''S$ denotes the image of S.)

Lemma 1

$i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding if and only if i satisfies (a), (b), (c) and

(d') *for all $q \in \mathbb{Q}$, there is $p \in \mathbb{P}$ s.t., for any $p' \leq_{\mathbb{P}} p$, $i(p') \not\perp_{\mathbb{Q}} q$.*

Proof

Lemma 2

Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. If H is a $(V, \mathbb{Q})$-generic filter, then $i^{-1}{}''H$ is a $(V, \mathbb{P})$-generic filter.

Proof

Two steps iteration

Suppose that $\mathbb{P}$ is a p.o. The following Lemma follows from the definition of $\mathbb{P}$-names and $\Vdash_{\mathbb{P}}$ "$\cdots$" (see Lemma 19):

Lemma 3

For a $\mathbb{P}$-name $\dot{x}$, there is a cardinal κ s.t., for any $\mathbb{P}$-name $\dot{y}$, there is $\dot{y}' \in V^{\mathbb{P}} \cap \mathcal{H}(\kappa)$ s.t. $\Vdash_{\mathbb{P}}$ "$\dot{x} \not\equiv \emptyset \rightarrow \dot{y}' \in \dot{x}$" and

$$\Vdash_{\mathbb{P}} \text{“} \dot{y} \in \dot{x} \rightarrow \dot{y} \equiv \dot{y}' \text{”}.$$

- Let $\kappa^{\mathbb{P}}_{\dot{x}}$ denote the smallest cardinal κ with the property above.
- For a p.o. $\mathbb{P}$ and a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ of a p.o., let $\mathbb{P} * \dot{\mathbb{Q}}$ be the p.o. defined by

$$\mathbb{P} * \dot{\mathbb{Q}} = \{\langle p, \dot{q}\rangle : p \in \mathbb{P},\ \dot{q} \in V^{\mathbb{P}} \cap \mathcal{H}(\kappa^{\mathbb{P}}_{\dot{\mathbb{Q}}}),\ \Vdash_{\mathbb{P}} \text{“} \dot{q} \in \dot{\mathbb{Q}} \text{”}\}$$

with the preorder and the designated maximal element:

$$\langle p, \dot{q}\rangle \le_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p', \dot{q}'\rangle \ \Leftrightarrow\ p \le_{\mathbb{P}} p' \text{ and } p \Vdash_{\mathbb{P}} \text{“} \dot{q} \le_{\dot{\mathbb{Q}}} \dot{q}' \text{”};$$

$$\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} = \langle \mathbb{1}_{\mathbb{P}}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}}\rangle.$$

- In the following, we drop the mention of "$\dot{q} \in V^{\mathbb{P}} \cap \mathcal{H}(\kappa^{\mathbb{P}}_{\dot{\mathbb{Q}}})$" for short.

Two steps iteration (2/3)

- For a p.o. $\mathbb{P}$ and a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ of a p.o., let $\mathbb{P} * \dot{\mathbb{Q}}$ be defined by:

  $$\mathbb{P} * \dot{\mathbb{Q}} = \{\langle p, \dot{q}\rangle \,:\, p \in \mathbb{P},\ \Vdash_{\mathbb{P}} \text{“}\dot{q} \in \dot{\mathbb{Q}}\text{”}\}$$

  with the preorder and the designated maximal element:

  $$\langle p, \dot{q}\rangle \leq_{\mathbb{P}*\dot{\mathbb{Q}}} \langle p', \dot{q}'\rangle \ \Leftrightarrow\ p \leq_{\mathbb{P}} p' \text{ and } p \Vdash_{\mathbb{P}} \text{“}\dot{q} \leq_{\dot{\mathbb{Q}}} \dot{q}'\text{”};$$

  $$\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} = \langle \mathbb{1}_{\mathbb{P}}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}}\rangle.$$

Lemma 4

$i : \mathbb{P} \to \mathbb{P} * \dot{\mathbb{Q}};\ p \mapsto \langle p, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}}\rangle$ *is a complete embedding.*

Proof

- The following is easy to prove:

Lemma 5

If G *is a* $(V, \mathbb{P})$*-generic filter and* H *is a* $(V[G], \dot{\mathbb{Q}}^G)$*-generic filter, then* $G \times H = \{\langle p, \dot{q}\rangle \,:\, p \in G,\ \dot{q}^G \in H\}$ *is a* $(V, \mathbb{P} * \dot{\mathbb{Q}})$*-generic filter and we have* $V[G][H] = V[G \times H]$.

- We also have the “converse” of Lemma 5.

Two steps iteration (3/3)

Lemma 5

If G is a $(V, \mathbb{P})$-generic filter and H is a $(V[G], \dot{\mathbb{Q}}^G)$-generic filter, then $G \times H = \{\langle p, \dot{q}\rangle : p \in G,\ \dot{q}^G \in H\}$ is a $(V, \mathbb{P} * \dot{\mathbb{Q}})$-generic filter and we have $V[G][H] = V[G \times H]$.

- We also have the "converse" of Lemma 5.

Lemma 6

If $\tilde{H}$ is a $(V, \mathbb{P} * \dot{\mathbb{Q}})$-generic filter, letting $G = i^{-1}{}''\tilde{H}$ for the complete embedding i as in Lemma 4, $\{\dot{q}^G : \langle p, \dot{q}\rangle \in \tilde{H}\}$ is a $(V[G], \dot{\mathbb{Q}}^G)$-generic filter. and $V[G][H] = V[\tilde{H}]$.

Proof

- By the lemmas above, we can capture all the features of two step iteration of forcing extensions by a single p.o. of the form $\mathbb{P} * \dot{\mathbb{Q}}$.

Two steps iteration with c.c.c. p.o.

Lemma 7

If $\mathbb{P}$ *is a c.c.c. p.o. and* $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ *is a c.c.c. p.o.*", *then* $\mathbb{P} * \dot{\mathbb{Q}}$ *is also a c.c.c. p.o..*

Proof

Names in a two steps iteration

- Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. We can define an embedding $i : V^{\mathbb{P}} \to V^{\mathbb{Q}}$ recursively by:

$$i(\dot{x}) = \{\langle i(\dot{y}), i(p)\rangle \ : \ \langle \dot{y}, p\rangle \in \dot{x}\}.$$

Lemma 8

Suppose that H is a $(V, \mathbb{Q})$-generic filter. For any $\mathbb{P}$-name $\dot{x}$, we have

$$\dot{x}^{i^{-1}{}''H} = i(\dot{x})^H$$

Proof. By the induction on $rank(\dot{x})$:

$$\dot{x}^{i^{-1}{}''H} = \{\dot{y}^{i^{-1}{}''H} \ : \ \langle \dot{y}, p\rangle \in \dot{x},\ p \in i^{-1}{}''H\}$$
$$= \{\imath(\dot{y})^H \ : \ \langle i(\dot{y}), i(p)\rangle \in i(\dot{x}),\ i(p) \in H\} = i(\dot{x})^H.$$

□

- In the (advanced) literature, $\dot{x}$ and $i(\dot{x})$ are very often identified without any caution.

Finite support iteration

"$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ is a finite support iteration of length γ"
is defined by induction on $\gamma \in \mathrm{ON}$ as follows:

(1) $\mathbb{P}_0 = \{\emptyset\}$ (the singleton of the sequence of length zero);

(2) $\mathbb{P}_\gamma$ consists of sequences of length γ;

(3) $\dot{\mathbb{Q}}_\beta = \langle \dot{\mathbb{Q}}_\beta, \dot{\leq}_{\dot{\mathbb{Q}}_\beta}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta} \rangle$ is a $\mathbb{P}_\beta$-name of a p.o.;

(4) If $\gamma = \delta + 1$ then $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ is a finite support iteration of length δ and $\mathbb{P}_\gamma = \{p^\frown\langle \dot{q} \rangle : \langle p, \dot{q} \rangle \in \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta\}$;

(4a) If $\gamma = \delta + 1$ and p, $p' \in \mathbb{P}_\gamma$,
$p \leq_{\mathbb{P}_\gamma} p' \;\Leftrightarrow\; \langle p \restriction \delta, p(\delta) \rangle \leq_{\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta} \langle p' \restriction \delta, p'(\delta) \rangle$;

(5) If γ is a limit ordinal, then $p \in \mathbb{P}_\gamma$ if and only if
$p \restriction (\alpha) \in \mathbb{P}_\alpha$ for all $\alpha < \gamma$, and
$\mathrm{supp}(p) = \{\alpha < \gamma : p(\alpha) \neq \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\alpha}\}$ is finite.

(5a) If γ is a limit ordinal and p, $p' \in \mathbb{P}_\gamma$, then $p \leq_{\mathbb{P}_\gamma} p' \;\Leftrightarrow$
$p \restriction (\alpha) \leq_{\mathbb{P}_\alpha} p' \restriction (\alpha)$ for all $\alpha < \gamma$.

Finite support iteration

(1) $\mathbb{P}_0 = \{\emptyset\}$ (the singleton of the sequence of length zero);

(2) $\mathbb{P}_\gamma$ consists of sequences of length γ;

(3) $\dot{\mathbb{Q}}_\beta = \langle \dot{\mathbb{Q}}_\beta, \dot{\leq}_{\dot{\mathbb{Q}}_\beta}, \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta} \rangle$ is a $\mathbb{P}_\beta$-name of a p.o.;

(4) If $\gamma = \delta + 1$ then $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \delta,\ \beta < \delta \rangle$ is a finite support iteration of length δ and $\mathbb{P}_\gamma = \{ p^\frown\langle \dot{q} \rangle : \langle p, \dot{q} \rangle \in \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta \}$;

(4a) If $\gamma = \delta + 1$ and p, $p' \in \mathbb{P}_\gamma$,
$p \leq_{\mathbb{P}_\gamma} p' \ \Leftrightarrow\ \langle p \restriction \delta, p(\delta) \rangle \leq_{\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta} \langle p' \restriction \delta, p'(\delta) \rangle$;

(5) If γ is a limit ordinal, for all $p \in \mathbb{P}_\gamma$ and $\alpha < \gamma$,
$p \restriction (\alpha) \in \mathbb{P}_\alpha$ and
$\mathrm{supp}(p) = \{ \alpha < \gamma : p(\alpha) \neq \dot{\mathbb{1}} \}_{\dot{\mathbb{Q}}_\alpha}$ is finite.

(5a) If γ is a limit ordinal and p, $p' \in \mathbb{P}_\gamma$, then $p \leq_{\mathbb{P}_\gamma} p' \ \Leftrightarrow$
$p \restriction (\alpha) \leq_{\mathbb{P}_\alpha} p' \restriction (\alpha)$ for all $\alpha < \gamma$.

(6) $\mathbb{P}_\alpha = \langle \mathbb{P}_\alpha, \leq_{\mathbb{P}_\alpha}, \vec{\mathbb{1}} \rangle$ where $\vec{\mathbb{1}} = \langle \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi} : \xi < \alpha \rangle$.

Finite support iteration (2/3)

The following can be proved by the induction on γ:

Lemma 9

Suppose that $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ *is a finite support iteration and* p, $p' \in \mathbb{P}_\gamma$. *Then*

$$p \leq_{\mathbb{P}_\gamma} p' \quad \Leftrightarrow$$
$$p \restriction (\alpha) \Vdash_{\mathbb{P}_\alpha} \text{“} p(\alpha) \leq_{\dot{\mathbb{Q}}_\alpha} p'(\alpha) \text{”} \text{ for all } \alpha < \gamma.$$

- For a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ of length γ, and $\alpha \leq \beta \leq \gamma$ let

$$i_{\alpha,\beta} : \mathbb{P}_\alpha \to \mathbb{P}_\beta;\ p \mapsto p ^\frown \vec{\mathbb{1}}, \text{ where } \vec{\mathbb{1}} = \langle \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\xi} : \alpha \leq \xi < \beta \rangle.$$

Lemma 10

For all $\alpha \leq \beta \leq \gamma$, $i_{\alpha,\beta}$ *is a complete embedding of* $\mathbb{P}_\alpha$ *into* $\mathbb{P}_\beta$.

Finite support iteration (3/3)

Lemma 10

For all $\alpha \leq \beta \leq \gamma$, $i_{\alpha,\beta}$ is a complete embedding of $\mathbb{P}_\alpha$ into $\mathbb{P}_\beta$.

Proof. By induction on β. At the successor step, use Lemma 4. □

- Suppose that G_γ is a $(V, \mathbb{P}_\gamma)$-generic filter. For $\alpha < \gamma$, let
$$G_\alpha = \{p \restriction (\alpha) : p \in G_\gamma\}.$$
- Note that $G_\alpha = (i_{\alpha,\gamma})^{-1}{''}G_\gamma$. Hence G_α is a $(V, \mathbb{P}_\alpha)$-generic filter by Lemma 2 and Lemma 10.
- By Lemma 2 and Lemma 10, G_α is a $(V, \mathbb{P}_\alpha)$-generic filter and
$$V \subseteq V[G_1] \subseteq V[G_2] \subseteq \cdots \subseteq V[G_\alpha] \subseteq \cdots \subseteq V[G_\gamma].$$
- By the definition of the successor step and Lemma 6, we have
$$V[G_{\alpha+1}] = V[G_\alpha][H_\alpha]$$
where H_α is a $(V[G_\alpha], (\dot{\mathbb{Q}}_\alpha)^{G_\alpha})$-generic filter.

Finite support iteration of c.c.c. p.o.s

Theorem 11

Suppose that $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ *is a finite support iteration s.t.*

$$\Vdash_{\mathbb{P}_\beta} \text{“}\dot{\mathbb{Q}}_\beta \textit{ is a c.c.c. p.o.''} \textit{ for all } \beta < \gamma.$$

Then $\mathbb{P}_\gamma$ *satisfies the c.c.c.*

Proof

Martin's Axiom

- For a p.o. $\mathbb{P}$ and a set $\mathcal{D}$ of dense subsets of $\mathbb{P}$, $G \subseteq \mathbb{P}$ is a **$\mathcal{D}$-generic filter** if G is a filter and $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.
- If $\mathcal{D}$ is countable then a $\mathcal{D}$-generic filter over $\mathbb{P}$ always exists.
- The following assertion is called **Martin's Axiom (MA)**:

> (MA) For any c.c.c. p.o. $\mathbb{P}$ and a set $\mathcal{D}$ of dense subsets of $\mathbb{P}$ of cardinality $< 2^{\aleph_0}$ there is a $\mathcal{D}$-generic filter over $\mathbb{P}$.

- By the remark above, CH implies MA.
- "$< 2^{\aleph_0}$" in the definition of MA cannot be replaced by $\leq 2^{\aleph_0}$.

Proof

Lemma 12

MA *is equivalent to the assertion:*

(*) *For any c.c.c. p.o.* $\mathbb{P}$ **of cardinality** $< 2^{\aleph_0}$ *and a set* $\mathcal{D}$ *of dense subsets of* $\mathbb{P}$ *of cardinality* $< 2^{\aleph_0}$ *there is a* $\mathcal{D}$*-generic filter over* $\mathbb{P}$.

Proof

MA $+ \neg$CH is consistent

Theorem 13 (Solovay and Tennenbaum, 1971(昭和 46))

Let κ be a regular cardinal $> \aleph_1$ s.t. $2^{<\kappa} = \kappa$. Then there is a c.c.c. p.o. $\mathbb{P}$ s.t.

$$\Vdash_{\mathbb{P}} \text{“}\mathsf{MA} + 2^{\aleph_0} = \kappa\text{”}$$

Proof (We need some preparation before beginning with the proof.)

Corollary 14

Let n be some (concretely) given number > 1. If ZFC is consistent, then so is ZFC $+$ MA $+ 2^{\aleph_0} = \aleph_n$.

Proof. Start from the Gödel's L (the class of constructible sets) which is a “model” of ZFC $+$ GCH. Then apply Theorem 13. □

An application of MA

- A linearly ordered set $S = \langle S, \leq_S \rangle$ is a **Souslin line** if S is densely ordered c.c.c. (i.e. every family of pairwise disjoint intervals is countable) and non separable (no countable dense subset).
- A tree T of height ω_1 is a **Souslin tree** if every branch of T is countable and every antichain (pairwise incomparable subset) is countable.

Lemma 15

A Souslin line can be constructed from a Souslin tree and vice verca.

Theorem 16 (R. Jensen)

Assume $V = L$. Then there is a Souslin tree (and hence there is a Souslin line).

An application of MA (2/2)

Theorem 17

Assume MA + ¬CH. Then there is no Souslin tree (and hence there is no Souslin line).

Proof

Corollary 18

The assertion "there is a Souslin line" is independent over ZFC.

Proof. By Theorem 16, Theorem 17 and Lemma 15. □

Three Lemmas for the proof of Theorem 13

- For a p.o. $\mathbb{P}$ and for a set a (in the ground model V), a **nice $\mathbb{P}$-name of a subset of a** is a $\mathbb{P}$-name $\dot{d}$ of the form

$$\dot{d} = \bigcup\{\{\check{c}\} \times A_c \,:\, c \in a\}$$

where each A_c is some antichain in $\mathbb{P}$.

Lemma 19

For any $\mathbb{P}$-name $\dot{b}$ there is a nice $\mathbb{P}$-name $\dot{d}$ of a suset of a s.t. $\Vdash_{\mathbb{P}}$ "$\dot{b} \subseteq \check{a} \rightarrow \dot{b} \equiv \dot{d}$".

Proof

- Note that there are class many $\mathbb{P}$-names but there are only set many nice $\mathbb{P}$-names of subsets of an a. In particular:

Lemma 20 ((A variation of) this implies Lemma 3.)

For a p.o. $\mathbb{P}$ and a set a, there are at most $2^{|\mathbb{P}|\cdot|a|}$ nice $\mathbb{P}$-names of subsets of a.

Proof

Three Lemmas for the proof of Theorem 13 (2/2)

Lemma 21

For any cardinal $\kappa \geq \aleph_0$ there is a mapping (the book-keeping)

$$bk : \kappa \to \kappa \times \kappa \ \alpha \mapsto \langle bk_0(\alpha), bk_1(\alpha)\rangle$$

s.t., (a) $bk_0(\alpha) = 0$ *or* $bk_0(\alpha) < \alpha$ *for all* $\alpha < \kappa$; (b) *For any* $\langle \beta, \gamma\rangle \in \kappa \times \kappa$ *there are* κ *many* $\alpha < \kappa$ *s.t.* $bk(\alpha) = \langle \beta, \gamma\rangle$.

Proof.

- Let $f : \kappa \to \kappa \times \kappa$; $\alpha \mapsto \langle f_0(\alpha), f_1(\alpha)\rangle$ be s.t.

  For any $\langle \beta, \gamma\rangle \in \kappa \times \kappa$ there are κ many $\alpha < \kappa$ s.t. $f(\alpha) = \langle \beta, \gamma\rangle$.

- Let $bk : \kappa \to \kappa \times \kappa$ be defined then by

$$bk(\alpha) = \begin{cases} f(\alpha); & \text{if } f_0(\alpha) = 0 \text{ or } f_0(\alpha) < \alpha, \\ \langle 0, 0\rangle; & \text{other wise} \end{cases}$$

  for $\alpha < \kappa$.

- This bk is then as desired. □

Proof of Theorem 13

- Suppose that GCH holds and let κ be a regular cardinal (acutually what we need is that κ is a regular cardinal with $2^{<\kappa} = \kappa$).
- We define inductively the FS-iteration

  $$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \;:\; \alpha \le \gamma,\ \beta < \kappa \rangle$$

  of c.c.c. p.o. (see Theorem 11) together with a sequence

  $$\langle \langle \dot{Q}_{\alpha,\gamma}, \dot{\le}_{\alpha,\gamma}, \dot{\mathbb{1}}_{\alpha,\gamma} \rangle \;:\; \alpha, \gamma < \kappa \rangle \text{ s.t.}$$

(a) $|\, P_\alpha \,| < \kappa$ for all $\alpha < \kappa$;

(b) (α) $\dot{Q}_{\alpha,\gamma}$ is a nice $\mathbb{P}_\alpha$-name of an infinite cardinal $< \kappa$;
(β) $\dot{\le}_{\alpha,\gamma}$ is a nice $\mathbb{P}_\alpha$-name of a preordering on $\dot{Q}_{\alpha,\gamma}$;
(γ) $\dot{\mathbb{1}}_{\alpha,\gamma}$ is a nice $\mathbb{P}_\alpha$-name of a maximal element of $\dot{Q}_{\alpha,\gamma}$
for all α, $\gamma < \kappa$.

(c) For each $\alpha < \kappa$, $\langle \langle \dot{Q}_{\alpha,\gamma}, \dot{\le}_{\alpha,\gamma}, \dot{\mathbb{1}}_{\alpha,\gamma} \rangle \;:\; \gamma < \kappa \rangle$ is an enumeration of the triples of the form as in (b).

Proof of Theorem 13 (2/3)

(a) $|P_\alpha| < \kappa$ for all $\alpha < \kappa$;

(b) (α) $\dot{Q}_{\alpha,\gamma}$ is a nice $\mathbb{P}_\alpha$-name of an infinite cardinal $< \kappa$;
(β) $\dot{\leq}_{\alpha,\gamma}$ is a nice $\mathbb{P}_\alpha$-name of a preordering on $\dot{Q}_{\alpha,\gamma}$;
(γ) $\dot{\mathbb{1}}_{\alpha,\gamma}$ is a nice $\mathbb{P}_\alpha$-name of a maximal element of $\dot{Q}_{\alpha,\gamma}$
for all α, $\gamma < \kappa$.

(c) For each $\alpha < \kappa$, $\langle\langle \dot{Q}_{\alpha,\gamma}, \dot{\leq}_{\alpha,\gamma}, \dot{\mathbb{1}}_{\alpha,\gamma}\rangle : \gamma < \kappa\rangle$ is an enumeration of the triples of the form as in (b).

- If $\langle \dot{Q}_{\alpha,\gamma}, \dot{\leq}_{\alpha,\gamma}, \dot{\mathbb{1}}_{\alpha,\gamma}\rangle$ as above is seen as a $\mathbb{P}_\alpha$-name of a p.o., we shall call it $\dot{\mathbb{Q}}_{\alpha,\gamma}$.

(d) $\dot{Q}_\alpha = i_{\beta,\gamma}(\dot{Q}_{\beta,\gamma})$, if $bk(\alpha) = \langle \beta, \gamma\rangle$ with $\beta < \alpha$ and $\Vdash_{\mathbb{P}}$ "$i_{\beta,\gamma}(\dot{Q}_{\beta,\gamma})$ is a $\mathbb{P}_\alpha$-name of a c.c.c. p.o.";
Otherwise $\dot{Q}_\alpha$ is a $\mathbb{P}_\alpha$-name of the trivial p.o..

Proof of Theorem 13 (3/3)

Exercises:

- ▷ Show that the p.o. $\mathbb{P}_\kappa$ constructed as above is as desired (Hint: show first that, if $\langle \dot{Q}_{\alpha,\gamma}, \dot{\leq}_{\alpha,\gamma}, \dot{\mathbb{1}}_{\alpha,\gamma} \rangle$ is a $\mathbb{P}_\kappa$-name of a c.c.c. p.o., then there are α, $\gamma < \kappa$ s.t. $\Vdash_{\mathbb{P}_\kappa}$ "$\langle i_{\alpha,\kappa}(\dot{Q}_{\alpha,\gamma}), i_{\alpha,\kappa}(\dot{\leq}_{\alpha,\gamma}), i_{\alpha,\kappa}(\dot{\mathbb{1}}_{\alpha,\gamma} \rangle) \cong \langle \dot{Q}_{\alpha,\gamma}, \dot{\leq}_{\alpha,\gamma}, \dot{\mathbb{1}}_{\alpha,\gamma} \rangle$").
- ▷ Check that the construction of $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ according to $(\mathrm{a})\tilde{}\ (\mathrm{d})$ is really possible. For example, we need some extra arguments to show that the condition (a) is preserved along with the inductive construction ((d) is used here): (a) is needed to enable the construction in (c)! — Lemma 20 is used here.
- ▷ Conclude that $\mathbb{P}_\kappa$ forces everything desired.

□ (Theorem 13)

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Proof of Lemma 1

- Suppose that i satisfies (a), (b) and (c) but not (d).

  Then there is a maxiamal antichain $A \subseteq \mathbb{P}$ s.t. $i''A$ is not a maximal antichain in $\mathbb{Q}$.

  Since $i''A$ is an antichain by (c), there is a $q_0 \in \mathbb{Q}$ s.t. $q_0 \perp_{\mathbb{Q}} q$ for all $q \in i''A$. For any $p \in \mathbb{P}$, there is a $p' \in A$ s.t. $p' \not\perp_{\mathbb{P}} p$ by the maximality of A. Let $p'' \leq_{\mathbb{P}} p, p'$. Then we have $p'' \leq_{\mathbb{P}} p$ and $i(p'') \perp_{\mathbb{Q}} q_0$. This shows that q_0 is a counterexample to (d').

- Suppose now that i satisfies (a), (b) and (c) but (d') does not hold and $q_0 \in \mathbb{Q}$ is a counterexample. Then $D = \{p' \in \mathbb{P} : i(p') \perp_{\mathbb{Q}} q_0\}$ is a dense subset in $\mathbb{P}$.

  Let A be a maximal antichain in D. Then, by the definition of D, elements of $i''A$ are incompatible with q_0 and hence $i''A$ is not a maximal antichain in $\mathbb{Q}$. Thus i does not satisfy (d).

q. e. d.

Proof of Lemma 2

- Let H be a $(V, \mathbb{Q})$-generic filter.
- Since it is clear that $i^{-1}{}''H$ is upward closed and pairwise compatible in $\mathbb{P}$, it is enough to show that, for a dense $D \subseteq \mathbb{P}$, we have $D \cap (i^{-1}{}''H) \neq \emptyset$.
- Suppose $D \subseteq \mathbb{P}$ is dense and let $A \subseteq D$ be a maximal antichain. Since i is a complete mapping, $i''A$ is a maximal antichain in $\mathbb{Q}$. It follows that

  $$D' = \{q \in \mathbb{Q} : q \leq_{\mathbb{Q}} i(a) \text{ for some } a \in A\}$$

  is dense in $\mathbb{Q}$.
- Let $q \in H \cap D'$ and let $a \in A$ be s.t. $q \leq_{\mathbb{Q}} i(a)$. Then, since H is a filter, we have $i(a) \in H$ and hence

  $$a \in (i^{-1}{}''H) \cap A \subseteq (i^{-1}{}''H) \cap D.$$

q. e. d.

Proof of Lemma 4

- It is clear that i satisfies (a), (b), (c) in the definition of the complete embedding.
- We show that i also satisfies (d'). Suppose $\langle p, \dot{q} \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$. For any $p' \leq_{\mathbb{P}} p$, since

$$\langle p', \dot{q} \rangle \leq_{\mathbb{P} * \dot{\mathbb{Q}}} \langle p, \dot{q} \rangle, \ \langle p', \dot{\mathbb{1}}_{\dot{\mathbb{Q}}} \rangle = i(p'),$$

we have $i(p') \not\perp_{\mathbb{P} * \dot{\mathbb{Q}}} \langle p, \dot{q} \rangle$.

q. e. d.

Proof of Lemma 6

- Suppose that $\tilde{H}$ is a $(V, \mathbb{P} * \dot{\mathbb{Q}})$-generic filter and let $G = i^{-1}{}''\tilde{H}$ for the mapping i as in Lemma 4.
- G is a $(V, \mathbb{P})$-generic filter by Lemma 2.
- We can show that $H = \{\dot{q}^G : \langle p, \dot{q} \rangle \in \tilde{H}\}$ is a $(V[G], \dot{\mathbb{Q}}^G)$-generic filter (Exercise!) and
- $V[G][H] = V[\tilde{H}]$ (Exercise!).

q. e. d.

Proof of Lemma 7

- We prove the Lemma 7 using the following (very powerful) general lemma on c.c.c. p.o.s.

Lemma A 1 (S. Shelah)

Suppose that $\mathbb{P}$ is a c.c.c. p.o. and $p_\xi \in \mathbb{P}$, $\xi < \omega_1$. Then

$$S = \{\beta < \omega_1 : p_\beta \Vdash_{\mathbb{P}} \text{“}\{\alpha \in \omega_1 : p_\alpha \in \dot{G}\} \textit{is stationary in } \omega_1\text{”}\}$$

contains a club set in ω_1.

Proof. Otherwise $S' = \omega_1 \setminus S$ is stationary. For $\beta \in S'$, let $q_\beta \leq_{\mathbb{P}} p_\beta$ and club $C_\beta \subseteq \omega_1$ (in the ground model!) be s.t.

$$q_\beta \Vdash_{\mathbb{P}} \text{“}\forall \alpha \in C_\beta (p_\alpha \notin \dot{G})\text{”}.$$

- Note that for all $\alpha \in C_\beta$, q_β and p_α are incompatible.
- Choose β_ξ, $\xi \in \omega_1$ inductively s.t. $\beta_\xi \in S' \cap \bigcap\{C_{\beta_\eta} : \eta < \xi\}$. Then $\{q_{\beta_\xi} : \xi < \omega_1\}$ is pairwise incompatible.
- This is a contradiction to the c.c.c. of $\mathbb{P}$. □

Proof of Lemma 7 (2/2)

- Suppose that $\langle p_\alpha, \dot{q}_\alpha \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$ for $\alpha \in \omega_1$.
- By Lemma A1, there is $\beta < \omega_1$ s.t.

  $$p_\beta \Vdash_{\mathbb{P}} \text{“}\{\alpha \in \omega_1 : p_\alpha \in \dot{G}\} \text{ is uncountable”}.$$
- Let $p \leq_{\mathbb{P}} p_\beta$ and $\alpha_0, \alpha_1 < \omega_1$ be s.t. $\alpha_0 \neq \alpha_1$,

  $$p \Vdash_{\mathbb{P}} \text{“}p_{\alpha_0}, p_{\alpha_1} \in \dot{G}\text{”}$$

  and $\dot{q}_{\alpha_0}$ and $\dot{q}_{\alpha_1}$ are compatible.

  (This is possible since $\Vdash_{\mathbb{P}}$ “$\dot{\mathbb{Q}}$ has the c.c.c.”.)
- Then $\langle p_{\alpha_0}, \dot{q}_{\alpha_0} \rangle$ and $\langle p_{\alpha_1}, \dot{q}_{\alpha_1} \rangle$ are compatible.

q. e. d.

Proof of Theorem 11

- We prove the theorem by induction on γ.
- If $\gamma = 0$, the assertion of the theorem is trivial.
- If $\gamma = \delta + 1$ and the theorem holds for δ, then we have that $\mathbb{P}_\delta$ is c.c.c. and $\Vdash_{\mathbb{P}_\delta}$ "$\dot{\mathbb{Q}}_\delta$ is a c.c.c. p.o.".
- Since $\mathbb{P}_\gamma \cong \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta$, it follows by Lemma 7 that $\mathbb{P}_\gamma$ also satisfies the c.c.c.
- Suppose now that γ is a limit ordinal and the theorem holds for all $\delta < \gamma$.
- Suppose that $p_\alpha \in \mathbb{P}_\gamma$, $\alpha < \omega_1$. We have to show that there are two distinct $\alpha_0, \alpha_1 < \omega_1$ s.t. $p_{\alpha_0} \not\perp_{\mathbb{P}_\gamma} p_{\alpha_1}$.

Proof of Theorem 11 (2/2)

- Suppose now that γ is a limit ordinal and the theorem holds for all $\delta < \gamma$.
- Suppose that $p_\alpha \in \mathbb{P}_\gamma$, $\alpha < \omega_1$. We have to show that there are two distinct $\alpha_0, \alpha_1 < \omega_1$ s.t. $p_{\alpha_0} \not\perp_{\mathbb{P}_\gamma} p_{\alpha_1}$.
- By the Δ-System Lemma, we can find an uncountable $I \subseteq \omega_1$ s.t. $\{\mathrm{supp}(p_\alpha) : \alpha \in I\}$ forms a Δ-system with the root r. Let $\gamma_0 = \max(r) + 1$. By the induction hypothesis $\{p_\alpha \restriction \gamma_0 : \alpha \in I\}$ is not an antichain in $\mathbb{P}_{\gamma_0}$. Thus there are $\alpha_0, \alpha_1 \in I$, $\alpha_0 \neq \alpha_1$ s.t. $p_{\alpha_0} \restriction \gamma_0$ and $p_{\alpha_1} \restriction \gamma_0$ are compatible. Let $p^* \in \mathbb{P}_{\gamma_0}$ be s.t. $p^* \leq_{\mathbb{P}_{\gamma_0}} p_{\alpha_0} \restriction \gamma_0, p_{\alpha_1} \restriction \gamma_0$.
  Let $p \in \mathbb{P}_\gamma$ be defined by
  $$p(\xi) = \begin{cases} p(\xi), & \text{if } \xi < \gamma_0; \\ p_{\alpha_0}(\xi), & \text{if } \xi \in \mathrm{supp}(p_{\alpha_0}) \setminus \gamma_0; \\ p_{\alpha_1}(\xi), & \text{otherwise} \end{cases}$$
  for $\xi < \omega_1$.
- Then we have $p \leq_{\mathbb{P}_\gamma} p_{\alpha_0}, p_{\alpha_1}$.

q. e. d.

Proof of Lemma 17

- Suppose that T is a Souslin tree. Without loss of generality, we may assume that, for any $t \in T$ and $\alpha < \omega_1$, there is $t' \in T$ s.t. $t \leq_T t'$ and $ht(t) \geq \alpha$.
- Let $\mathbb{P}_T$ be the p.o. defined by $\mathbb{P}_T = T$, $t \leq_{\mathbb{P}_T} t' \Leftrightarrow t \geq_T t'$ for all $t, t' \in T$ and $\mathbb{1}_{\mathbb{P}_T} =$ the root of T.
- $\mathbb{P}_T$ has the c.c.c.

  Let $D_\alpha = \{t \in T : ht(t) \geq \alpha\}$. Then D_α is dense in $\mathbb{P}_T$ for all $\alpha < \omega_1$. Let $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$. By $\mathrm{MA} + \neg\mathrm{CH}$, there is a $\mathcal{D}$-generic filter G over $\mathbb{P}_T$.
- By the $\mathcal{D}$-genericity, G generates an uncountable branch of T. This is a contradiction to the assumption that T is a Souslin tree.

q. e. d.

Proof of Lemma 12

- It is clear that $(*)$ follows from MA.
- Assume that $(*)$ holds. Let $\mathbb{P}$ be a c.c.c. p.o. and $\mathcal{D}$ a family of dense subsets of $\mathbb{P}$ with $|\,\mathcal{D}\,| < 2^{\aleph_0}$.
- Let θ be a sufficiently large regular cardinal and let $M \prec \mathcal{H}(\theta)$ be s.t. $\mathbb{P}$, $\mathcal{D} \in M$, $\mathcal{D} \subseteq M$ and $|\,M\,| < 2^{\aleph_0}$.
- Then $\mathbb{P}_0 = \langle \mathbb{P} \cap M, \leq_{\mathbb{P}} \cap (\mathbb{P} \cap M)^2, \mathbb{1}_{\mathbb{P}} \rangle$ is a c.c.c. p.o. of cardinality $< 2^{\aleph_0}$ and $\mathcal{D}_0 = \{D \cap M \,:\, D \in \mathcal{D}\}$ is a family of dense subsets of $\mathbb{P}_0$ with $|\,\mathcal{D}_0\,| < 2^{\aleph_0}$.
- By $(*)$, there is a $\mathcal{D}_0$-generic filter G_0 on $\mathbb{P}_0$.
- $G = \{q \in \mathbb{P} \,:\, p \leq_{\mathbb{P}} q \text{ for some } p \in G_0\}$ is then a $\mathcal{D}$-generic filter on $\mathbb{P}$.

q. e. d.

Proof of the remark on p. 16

Lemma A 2

For a p.o. $\mathbb{P}$ adding a new real (i.e. a new subset of ω), the assertion

> *For any set $\mathcal{D}$ of dense subsets of $\mathbb{P}$ of cardinality $\leq 2^{\aleph_0}$ there is a $\mathcal{D}$-generic filter over $\mathbb{P}$.*

is incosistent.

Proof. Suppose that $\dot{a}$ is a $\mathbb{P}$-name of a new real. Let $\mathcal{P}(\omega) = \{a_\alpha : \alpha < 2^\omega\}$. For each $\alpha < 2^\omega$, let

$$D_\alpha = \{p \in \mathbb{P} : p \Vdash_{\mathbb{P}} \text{“} n \in \dot{a} \setminus \check{a}_\alpha \text{”} \text{ or } p \Vdash_{\mathbb{P}} \text{“} n \in \check{a}_\alpha \setminus \dot{a} \text{”} \text{ for some } n \in \omega\}.$$

Then D_α is dense in $\mathbb{P}$. $\mathcal{D} = \{D_\alpha : \alpha < 2^\omega\}$ has cardinality $\leq 2^{\aleph_0}$. But there is no $\mathcal{D}$-generic filter G, since for such a filter G, $\dot{a}^G = \{n \in \omega : p \Vdash_{\mathbb{P}} \text{“} \check{n} \in \dot{a} \text{”} \text{ for some } p \in G\}$ should be a real different from all a_α, $\alpha < 2^\omega$. *q. e. d.*

- If there is no Souslin tree then all non trivial c.c.c. p.o. add a new real.