# Set-theoretic aspects of topological reflection theorems

Sakaé Fuchino (渕野 昌)

Kobe University (神戸大学大学院 システム情報学研究科)

fuchino@diamond.kobe-u.ac.jp

http://kurt.scitec.kobe-u.ac.jp/~fuchino/

(December 22, 2010 (01:41 JST) version)

2010 年度 General Topology Symposium での講演
於 筑波大学

December 20, 2010

This presentation is typeset by pLATEX with beamer class.

# Coauthors



- The following results are obtained in joint researches mainly with:
  - **Assaf Rinot** (Ben Gurion University, B'er Sheba, Israel)
  - **Hiroshi Sakai** (酒井拓史, 神戸大学)
  - **Lajos Soukup** (Hungarian Academy of Science, Budapest, Hungary)
  - **Toshimichi Usuba** (薄葉季路，名古屋大学)

[1] S.F., **István Juhász**, Lajos Soukup, **Zoltán Szentmiklóssy** and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications Vol.157, 8 (June 2010), Special Issue dedicated to the Proceedings of the Conference "Advances in Set-Theoretic Topology" (in Honour of Tsugunori Nogura on his 60th Birthday), 1415-1429.

[2] S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba, More about Fodor-type Reflection Principle, preprint.

[3] S.F., Assaf Rinot, Openly generated Boolean algebras and the Fodor-type Reflection Principle, to appear in Fund. Math.

## Coauthors



- The following results are obtained in joint researches mainly with:
  - **Assaf Rinot** (Ben Gurion University, B'er Sheba, Israel)
  - **Hiroshi Sakai** (酒井拓史, 神戸大学)
  - **Lajos Soukup** (Hungarian Academy of Science, Budapest, Hungary)
  - **Toshimichi Usuba** (薄葉季路, 名古屋大学)


[1] S.F., **István Juhász**, Lajos Soukup, **Zoltán Szentmiklóssy** and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications Vol.157, 8 (June 2010), Special Issue dedicated to the Proceedings of the Conference "Advances in Set-Theoretic Topology" (in Honour of Tsugunori Nogura on his 60th Birthday), 1415-1429.

[2] S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba, More about Fodor-type Reflection Principle, preprint.

[3] S.F., Assaf Rinot, Openly generated Boolean algebras and the Fodor-type Reflection Principle, to appear in Fund. Math.

# Coauthors



- The following results are obtained in joint researches mainly with:
  - **Assaf Rinot** (Ben Gurion University, B'er Sheba, Israel)
  - **Hiroshi Sakai** (酒井拓史, 神戸大学)
  - **Lajos Soukup** (Hungarian Academy of Science, Budapest, Hungary)
  - **Toshimichi Usuba** (薄葉季路, 名古屋大学)

[1] S.F., **István Juhász**, Lajos Soukup, **Zoltán Szentmiklóssy** and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications Vol.157, 8 (June 2010), Special Issue dedicated to the Proceedings of the Conference "Advances in Set-Theoretic Topology" (in Honour of Tsugunori Nogura on his 60th Birthday), 1415-1429.

[2] S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba, More about Fodor-type Reflection Principle, preprint.

[3] S.F., Assaf Rinot, Openly generated Boolean algebras and the Fodor-type Reflection Principle, to appear in Fund. Math.

# Coauthors



- The following results are obtained in joint researches mainly with:
  - **Assaf Rinot** (Ben Gurion University, B'er Sheba, Israel)
  - **Hiroshi Sakai** (酒井拓史, 神戸大学)
  - **Lajos Soukup** (Hungarian Academy of Science, Budapest, Hungary)
  - **Toshimichi Usuba** (薄葉季路, 名古屋大学)

[1] S.F., **István Juhász**, Lajos Soukup, **Zoltán Szentmiklóssy** and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications Vol.157, 8 (June 2010), Special Issue dedicated to the Proceedings of the Conference "Advances in Set-Theoretic Topology" (in Honour of Tsugunori Nogura on his 60th Birthday), 1415-1429.

[2] S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba, More about Fodor-type Reflection Principle, preprint.

[3] S.F., Assaf Rinot, Openly generated Boolean algebras and the Fodor-type Reflection Principle, to appear in Fund. Math.

# Reflection theorems on metrizability



**Theorem 1 (A. Dow, 1988)**

*For a countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

- Is this theorem true for **locally** compact spaces ?
- (Folklore) Under $\square_{\aleph_1}$ there is a locally countably compact non metrizable space $X$ of cardinality $\aleph_2$ s.t. all $Y \in [X]^{\leq \aleph_1}$ are metrizable.

# Reflection theorems on metrizability

**Theorem 1 (A. Dow, 1988)**

*For a countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

- Is this theorem true for **locally** compact spaces ?
- (Folklore) Under $\square_{\aleph_1}$ there is a locally countably compact non metrizable space $X$ of cardinality $\aleph_2$ s.t. all $Y \in [X]^{\leq \aleph_1}$ are metrizable.

# Reflection theorems on metrizability



**Theorem 1 (A. Dow, 1988)**

*For a countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

- Is this theorem true for **locally** compact spaces ?

- (Folklore) Under $\square_{\aleph_1}$ there is a locally countably compact non metrizable space $X$ of cardinality $\aleph_2$ s.t. all $Y \in [X]^{\leq \aleph_1}$ are metrizable.

## Reflection theorems on metrizability



**Theorem 1 (A. Dow, 1988)**

*For a countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

- Is this theorem true for **locally** compact spaces ?
- (Folklore) Under $\Box_{\aleph_1}$ there is a locally countably compact non metrizable space $X$ of cardinality $\aleph_2$ s.t. all $Y \in [X]^{\leq \aleph_1}$ are metrizable.

# Reflection theorems on metrizability



**Theorem 1 (A. Dow, 1988)**

*For a countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

**Theorem 2 (Balogh, 2002)**

*Assume* **Axiom R** *. Then, for a* **locally** *countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

# Reflection theorems on metrizability



**Theorem 1 (A. Dow, 1988)**

*For a countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

**Theorem 2 (Balogh, 2002)**

*Assume* **Axiom R**. *Then, for a* **locally** *countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

## Reflection theorems on metrizability



**Theorem 2 (Balogh, 2002)**

*Assume* **Axiom R**. *Then, for a* **locally** *countably compact topological space* $X$,

*if all subspaces* $Y$ *of* $X$ *cardinality* $\leq \aleph_1$ *are metrizable then* $X$ *itself is metrizable.*

# Reflection theorems on metrizability



**Theorem 3 (S.F., Juhász, Soukup, Szentmiklóssy and Usuba[1], 2010)**

*Assume **FRP**. Then, for a **locally** countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

**Corollary 4**

*The reflection theorem above is consistent with arbitrarily large size of the continuum (and/or with MA).*

## Reflection theorems on metrizability



**Theorem 3 (S.F., Juhász, Soukup, Szentmiklóssy and Usuba[1], 2010)**

*Assume **FRP**. Then, for a **locally** countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

**Corollary 4**

*The reflection theorem above is consistent with arbitrarily large size of the continuum (and/or with MA).*

# Characterization of the reflection

Actually, Theorem 3 is optimal in the following sense:

**Theorem 5** (S.F., Sakai, Soukup and Usuba, [2], preprint)

*FRP is* **equivalent** *to the following assertion over ZFC:*

*For any locally countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

For "$\Leftarrow$" assume that FRP does not hold.

Then there are a regular $\kappa > \omega_1$, a stationary set $S \subseteq E^\kappa_\omega$ and a ladder system $\langle a_\alpha : \alpha \in S\rangle$ s.t. for all $\beta < \kappa$ there is a regressive $f : S \cap \beta \to \beta$ s.t.

(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

## Characterization of the reflection



Actually, Theorem 3 is optimal in the following sense:

**Theorem 5 (S.F., Sakai, Soukup and Usuba, [2], preprint)**

FRP *is* **equivalent** *to the following assertion over* ZFC*:*

*For any locally countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

For "$\Leftarrow$" assume that FRP does not hold.

Then there are a regular $\kappa > \omega_1$, a stationary set $S \subseteq E^\kappa_\omega$ and a ladder system $\langle a_\alpha : \alpha \in S \rangle$ s.t. for all $\beta < \kappa$ there is a regressive $f : S \cap \beta \to \beta$ s.t.

(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

# Characterization of the reflection

Actually, Theorem 3 is optimal in the following sense:

**Theorem 5** (S.F., Sakai, Soukup and Usuba, [2], preprint)

FRP *is* **equivalent** *to the following assertion over ZFC:*

*For any locally countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

For "$\Leftarrow$" assume that FRP does not hold.
Then there are a regular $\kappa > \omega_1$, a stationary set $S \subseteq E^{\kappa}_{\omega}$ and a ladder system $\langle a_\alpha : \alpha \in S \rangle$ s.t. for all $\beta < \kappa$ there is a regressive $f : S \cap \beta \to \beta$ s.t.

(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

## Characterization of the reflection



Actually, Theorem 3 is optimal in the following sense:

**Theorem 5** (S.F., Sakai, Soukup and Usuba, [2], preprint)

FRP *is* **equivalent** *to the following assertion over* ZFC*:*

*For any locally countably compact topological space $X$,*

*if all subspaces $Y$ of $X$ cardinality $\leq \aleph_1$ are metrizable then $X$ itself is metrizable.*

Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

For "$\Leftarrow$" assume that FRP does not hold.

Then there are a regular $\kappa > \omega_1$, a stationary set $S \subseteq E^\kappa_\omega$ and a ladder system $\langle a_\alpha : \alpha \in S \rangle$ s.t. for all $\beta < \kappa$ there is a regressive $f : S \cap \beta \rightarrow \beta$ s.t.

(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

# Characterization of the reflection



Actually, Theorem 3 is optimal in the following sense:

**Theorem 5** (S.F., Sakai, Soukup and Usuba, [2], preprint)

FRP *is* **equivalent** *to the following assertion over* ZFC*:*

*For any locally countably compact topological space* $X$,

*if all subspaces* $Y$ *of* $X$ *cardinality* $\leq \aleph_1$ *are metrizable then* $X$ *itself is metrizable.*

Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

For "$\Leftarrow$" assume that FRP does not hold.
Then there are a regular $\kappa > \omega_1$, a stationary set $S \subseteq E^\kappa_\omega$ and a ladder system $\langle a_\alpha : \alpha \in S \rangle$ s.t. for all $\beta < \kappa$ there is a regressive $f : S \cap \beta \to \beta$ s.t.

(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

# Proof of Theorem 5



Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

For "$\Leftarrow$" assume that FRP does not hold.
Then there are a regular $\kappa > \omega_1$, a stationary set $S \subseteq E^\kappa_\omega$ and a ladder system $\langle a_\alpha : \alpha \in S \rangle$ s.t. for all $\beta < \kappa$ there is a regressive $f : S \cap \beta \to \beta$ s.t.

(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

Without loss of generality we may assume that $a_\alpha \subseteq \kappa \setminus Lim(\kappa)$ for all $\alpha \in E$.

Let $X = E^\kappa_\omega \cup (\kappa \setminus Lim(\kappa))$ be with the Mrowká topology w.r.t. the ladder system $\langle a_\alpha : \alpha \in S \rangle$.

Then $X$ is locally compact and non-metrizable (by Fodor's Lemma).

But all subspaces $Y$ of $X$ of cardinality $< \kappa$ are metrizable (use $f$ as above for $\beta = \sup Y$ with (*)). $\square$ (**Theorem 5**)

## Proof of Theorem 5



Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

For "$\Leftarrow$" assume that FRP does not hold.
Then there are a regular $\kappa > \omega_1$, a stationary set $S \subseteq E^\kappa_\omega$ and a ladder system $\langle a_\alpha : \alpha \in S \rangle$ s.t. for all $\beta < \kappa$ there is a regressive $f : S \cap \beta \to \beta$ s.t.

(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

Without loss of generality we may assume that $a_\alpha \subseteq \kappa \setminus Lim(\kappa)$ for all $\alpha \in E$.

Let $X = E^\kappa_\omega \cup (\kappa \setminus Lim(\kappa))$ be with the Mrowká topology w.r.t. the ladder system $\langle a_\alpha : \alpha \in S \rangle$.

Then $X$ is locally compact and non-metrizable (by Fodor's Lemma).

But all subspaces $Y$ of $X$ of cardinality $< \kappa$ are metrizable (use $f$ as above for $\beta = \sup Y$ with (*)). □ (**Theorem 5**)

# Proof of Theorem 5



Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

For "$\Leftarrow$" assume that FRP does not hold.
Then there are a regular $\kappa > \omega_1$, a stationary set $S \subseteq E^\kappa_\omega$ and a ladder system $\langle a_\alpha : \alpha \in S \rangle$ s.t. for all $\beta < \kappa$ there is a regressive $f : S \cap \beta \to \beta$ s.t.

(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

Without loss of generality we may assume that $a_\alpha \subseteq \kappa \setminus Lim(\kappa)$ for all $\alpha \in E$.

Let $X = E^\kappa_\omega \cup (\kappa \setminus Lim(\kappa))$ be with the Mrowká topology w.r.t. the ladder system $\langle a_\alpha : \alpha \in S \rangle$.

Then $X$ is locally compact and non-metrizable (by Fodor's Lemma).

But all subspaces $Y$ of $X$ of cardinality $< \kappa$ are metrizable (use $f$ as above for $\beta = \sup Y$ with (*)). □ (**Theorem 5**)

# Proof of Theorem 5



Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

For "$\Leftarrow$" assume that FRP does not hold.
Then there are a regular $\kappa > \omega_1$, a stationary set $S \subseteq E^\kappa_\omega$ and a ladder system $\langle a_\alpha : \alpha \in S \rangle$ s.t. for all $\beta < \kappa$ there is a regressive $f : S \cap \beta \to \beta$ s.t.

(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

Without loss of generality we may assume that $a_\alpha \subseteq \kappa \setminus Lim(\kappa)$ for all $\alpha \in E$.

Let $X = E^\kappa_\omega \cup (\kappa \setminus Lim(\kappa))$ be with the Mrowká topology w.r.t. the ladder system $\langle a_\alpha : \alpha \in S \rangle$.

Then $X$ is locally compact and non-metrizable (by Fodor's Lemma).

But all subspaces $Y$ of $X$ of cardinality $< \kappa$ are metrizable (use $f$ as above for $\beta = \sup Y$ with (*)). □ (**Theorem 5**)

# Proof of Theorem 5



Sketch of the proof: "$\Rightarrow$" is just Theorem 3.

For "$\Leftarrow$" assume that FRP does not hold.
Then there are a regular $\kappa > \omega_1$, a stationary set $S \subseteq E^\kappa_\omega$ and a ladder system $\langle a_\alpha : \alpha \in S \rangle$ s.t. for all $\beta < \kappa$ there is a regressive $f : S \cap \beta \to \beta$ s.t.

(*) $a_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ is pairwise disjoint.

Without loss of generality we may assume that $a_\alpha \subseteq \kappa \setminus Lim(\kappa)$ for all $\alpha \in E$.

Let $X = E^\kappa_\omega \cup (\kappa \setminus Lim(\kappa))$ be with the Mrowká topology w.r.t. the ladder system $\langle a_\alpha : \alpha \in S \rangle$.

Then $X$ is locally compact and non-metrizable (by Fodor's Lemma).

But all subspaces $Y$ of $X$ of cardinality $< \kappa$ are metrizable (use $f$ as above for $\beta = \sup Y$ with (*)). □ (**Theorem 5**)

# Further reflection theorems (1/2)



FRP is also equivalent to the following topological reflection theorems:

For a locally separable, countably tight space $X$, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are meta-Lindelöf then $X$ itself is meta-Lindelöf.

(S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba [2])

▶ The reflection of metrizability of Theorem 3 or Theorem 5 is actually a corollary of the assertion above.

For a $T_1$ space with point countable base, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are left-separated then $X$ itself is left-separated.

([4] S.F., Left-separated topological spaces under Fodor-type Reflection Principle, RIMS Kokyuroku No.1619 (2008), 32-42.)

▶ W. Fleissner (1986) proved that the assertion above follows from Axiom R and it is refuted under the negation of the square principle.

## Further reflection theorems (1/2)



FRP is also equivalent to the following topological reflection theorems:

| |
|---|
| For a locally separable, countably tight space $X$, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are meta-Lindelöf then $X$ itself is meta-Lindelöf. |

(S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba [2])

- The reflection of metrizability of Theorem 3 or Theorem 5 is actually a corollary of the assertion above.

| |
|---|
| For a $T_1$ space with point countable base, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are left-separated then $X$ itself is left-separated. |

([4] S.F., Left-separated topological spaces under Fodor-type Reflec -tion Principle, RIMS Kokyuroku No.1619 (2008), 32-42.)

- W. Fleissner (1986) proved that the assertion above follows from Axiom R and it is refuted under the negation of the square principle.

# Further reflection theorems (1/2)



FRP is also equivalent to the following topological reflection theorems:

For a locally separable, countably tight space $X$, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are meta-Lindelöf then $X$ itself is meta-Lindelöf.

(S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba [2])

- The reflection of metrizability of Theorem 3 or Theorem 5 is actually a corollary of the assertion above.

For a $T_1$ space with point countable base, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are left-separated then $X$ itself is left-separated.

([4] S.F., Left-separated topological spaces under Fodor-type Reflection Principle, RIMS Kokyuroku No.1619 (2008), 32-42.)

- W. Fleissner (1986) proved that the assertion above follows from Axiom R and it is refuted under the negation of the square principle.

## Further reflection theorems (1/2)



FRP is also equivalent to the following topological reflection theorems:

For a locally separable, countably tight space $X$, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are meta-Lindelöf then $X$ itself is meta-Lindelöf.

(S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba [2])

- The reflection of metrizability of Theorem 3 or Theorem 5 is actually a corollary of the assertion above.

For a $T_1$ space with point countable base, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are left-separated then $X$ itself is left-separated.

([4] S.F., Left-separated topological spaces under Fodor-type Reflection Principle, RIMS Kokyuroku No.1619 (2008), 32-42.)

- W. Fleissner (1986) proved that the assertion above follows from Axiom R and it is refuted under the negation of the square principle.

# Further reflection theorems (1/2)



FRP is also equivalent to the following topological reflection theorems:

For a locally separable, countably tight space $X$, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are meta-Lindelöf then $X$ itself is meta-Lindelöf.

(S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba [2])

- The reflection of metrizability of Theorem 3 or Theorem 5 is actually a corollary of the assertion above.

For a $T_1$ space with point countable base, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are left-separated then $X$ itself is left-separated.

([4] S.F., Left-separated topological spaces under Fodor-type Reflection Principle, RIMS Kokyuroku No.1619 (2008), 32-42.)

- W. Fleissner (1986) proved that the assertion above follows from Axiom R and it is refuted under the negation of the square principle.

# Further reflection theorems (2/2)



FRP is also equivalent to the following topological reflection theorem:

For a countably tight space of local density $\aleph_1$, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are collectionwise Hausdorff then $X$ itself is collectionwise Hausdorff.

(S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba [2])

- Fleissner (1986) proved also the assertion above under Axiom R.

# Further reflection theorems (2/2)



FRP is also equivalent to the following topological reflection theorem:

For a countably tight space of local density $\aleph_1$, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are collectionwise Hausdorff then $X$ itself is collectionwise Hausdorff.

(S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba [2])

- Fleissner (1986) proved also the assertion above under Axiom R.

# Further reflection theorems (2/2)



FRP is also equivalent to the following topological reflection theorem:

> For a countably tight space of local density $\aleph_1$, if all subspaces $Y$ of $X$ of cardinality $\leq \aleph_1$ are collectionwise Hausdorff then $X$ itself is collectionwise Hausdorff.

(S.F., Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba [2])

- Fleissner (1986) proved also the assertion above under Axiom R.

# Reflection of openly generatedness

**Theorem 6 (S.F. and Rinot [3], to appear)**

*The following assertion is equivalent to* FRP *over* ZFC*:*

*A Boolean algebra $B$ is openly generated if and only if $\{C \in [B]^{<\aleph_2} : C \text{ is projective}\}$ contains a club subset.*

- S.F.(1994) proved the assertion above under Axiom R.
- Ingredients of the proof:
  - S. Koppelberg's theory of projective Boolean algebras
  - Freese-Nation property and weak Freese-Nation property
  - Bockstein separation property
  - $\omega$-stability of structures
  - FRP implies **Shelah's Strong Hypothesis (SSH)**

# Reflection of openly generatedness

**Theorem 6 (S.F. and Rinot [3], to appear)**

*The following assertion is equivalent to* FRP *over* ZFC*:*

*A Boolean algebra $B$ is openly generated if and only if $\{C \in [B]^{<\aleph_2} : C$ is projective$\}$ contains a club subset.*

- S.F.(1994) proved the assertion above under Axiom R.
- Ingredients of the proof:
  - S. Koppelberg's theory of projective Boolean algebras
  - Freese-Nation property and weak Freese-Nation property
  - Bockstein separation property
  - $\omega$-stability of structures
  - FRP implies **Shelah's Strong Hypothesis (SSH)**

# Reflection of openly generatedness

**Theorem 6 (S.F. and Rinot [3], to appear)**

*The following assertion is equivalent to* FRP *over* ZFC*:*

*A Boolean algebra $B$ is openly generated if and only if $\{C \in [B]^{<\aleph_2} : C \text{ is projective}\}$ contains a club subset.*

- S.F.(1994) proved the assertion above under Axiom R.
- Ingredients of the proof:
  - S. Koppelberg's theory of projective Boolean algebras
  - Freese-Nation property and weak Freese-Nation property
  - Bockstein separation property
  - $\omega$-stability of structures
  - FRP implies **Shelah's Strong Hypothesis (SSH)**

## Reflection of openly generatedness



**Theorem 6 (S.F. and Rinot [3], to appear)**

*The following assertion is equivalent to* FRP *over* ZFC:

*A Boolean algebra* $B$ *is openly generated if and only if* $\{C \in [B]^{<\aleph_2} : C \text{ is projective}\}$ *contains a club subset.*

- S.F.(1994) proved the assertion above under Axiom R.
- Ingredients of the proof:
  - S. Koppelberg's theory of projective Boolean algebras
  - Freese-Nation property and weak Freese-Nation property
  - Bockstein separation property
  - $\omega$-stability of structures
  - FRP implies **Shelah's Strong Hypothesis (SSH)**

## Reflection of openly generatedness



**Theorem 6 (S.F. and Rinot [3], to appear)**

*The following assertion is equivalent to* FRP *over* ZFC*:*

*A Boolean algebra $B$ is openly generated if and only if $\{C \in [B]^{<\aleph_2} : C \text{ is projective}\}$ contains a club subset.*

- S.F.(1994) proved the assertion above under Axiom R.
- Ingredients of the proof:
  - S. Koppelberg's theory of projective Boolean algebras
  - Freese-Nation property and weak Freese-Nation property
  - Bockstein separation property
  - $\omega$-stability of structures
  - FRP implies **Shelah's Strong Hypothesis (SSH)**

# Reflection of openly generatedness



**Theorem 6 (S.F. and Rinot [3], to appear)**

*The following assertion is equivalent to* FRP *over* ZFC*:*

*A Boolean algebra* $B$ *is openly generated if and only if* $\{C \in [B]^{<\aleph_2} : C \text{ is projective}\}$ *contains a club subset.*

- S.F.(1994) proved the assertion above under Axiom R.
- Ingredients of the proof:
  - S. Koppelberg's theory of projective Boolean algebras
  - Freese-Nation property and weak Freese-Nation property
  - Bockstein separation property
  - $\omega$-stability of structures
  - FRP implies **Shelah's Strong Hypothesis (SSH)**

# Reflection of openly generatedness

**Theorem 6 (S.F. and Rinot [3], to appear)**

*The following assertion is equivalent to* FRP *over* ZFC*:*

*A Boolean algebra $B$ is openly generated if and only if $\{C \in [B]^{<\aleph_2} : C \text{ is projective}\}$ contains a club subset.*

- S.F.(1994) proved the assertion above under Axiom R.
- Ingredients of the proof:
  - S. Koppelberg's theory of projective Boolean algebras
  - Freese-Nation property and weak Freese-Nation property
  - Bockstein separation property
  - $\omega$-stability of structures
  - FRP implies **Shelah's Strong Hypothesis (SSH)**

# Reflection of openly generatedness



**Theorem 6 (S.F. and Rinot [3], to appear)**

*The following assertion is equivalent to* FRP *over* ZFC*:*

*A Boolean algebra $B$ is openly generated if and only if $\{C \in [B]^{<\aleph_2} : C \text{ is projective}\}$ contains a club subset.*

- S.F.(1994) proved the assertion above under Axiom R.
- Ingredients of the proof:
  - S. Koppelberg's theory of projective Boolean algebras
  - Freese-Nation property and weak Freese-Nation property
  - Bockstein separation property
  - $\omega$-stability of structures
  - FRP implies **Shelah's Strong Hypothesis (SSH)**

# Reflection of openly generatedness

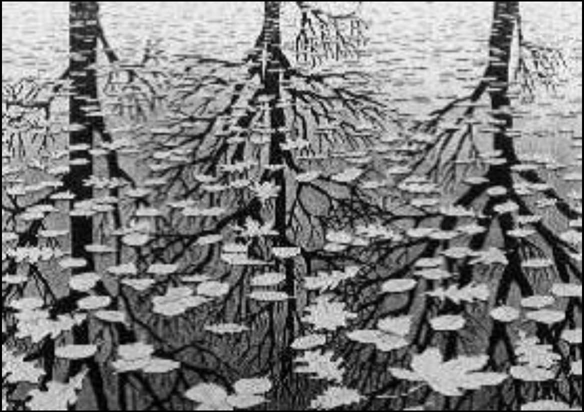
**Theorem 6 (S.F. and Rinot [3], to appear)**

*The following assertion is equivalent to* FRP *over* ZFC*:*

*A Boolean algebra $B$ is openly generated if and only if $\{C \in [B]^{<\aleph_2} : C$ is projective$\}$ contains a club subset.*

- S.F.(1994) proved the assertion above under Axiom R.
- Ingredients of the proof:
  - S. Koppelberg's theory of projective Boolean algebras
  - Freese-Nation property and weak Freese-Nation property
  - Bockstein separation property
  - $\omega$-stability of structures
  - FRP implies **Shelah's Strong Hypothesis (SSH)**

# ¡Gracias por su atención!



A part of: M.C. Escher, "Three Worlds" (1955)

These slides and their printer friendly version as well as the present versions of [1] $\sim$ [4] are downloadable at:
http://kurt.scitec.kobe-u.ac.jp/~fuchino/

# Axiom R



- **Axiom R** is the principle asserting that the following $\mathrm{AR}([\kappa]^{\aleph_0})$ holds for all cardinal $\kappa \geq \aleph_2$:

| |
|---|
| $(\mathrm{AR}([\kappa]^{\aleph_0}))$ For any stationary $S \subseteq [\kappa]^{\aleph_0}$ and a $\omega_1$-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, there is $I \in \mathcal{T}$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$. |

- For a set $X$ and cardinal $\kappa$, $[\mathbf{X}]^{\kappa} = \{x \in \mathcal{P}(X) : |x| = \kappa\}$. $[X]^{<\kappa}$ and $[X]^{\leq\kappa}$ are defined similarly.

- $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$ is said to be $\omega_1$**-club** if it is unbounded w.r.t. $\subseteq$ and closed w.r.t. union of $\subseteq$-chain of length $\omega_1$.

# Fodor-type Reflection Principle



- **Fodor-type Reflection Principle (FRP)** is the principle asserting that the following $\mathrm{FRP}(\kappa)$ holds for all regular cardinal $\kappa \geq \aleph_2$:

**$(\mathrm{FRP}(\kappa))$:** *For any stationary $S \subseteq E^\kappa_\omega$ and $g : S \to [\kappa]^{\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that*

- $\mathrm{cf}(I) = \omega_1$; $g(\alpha) \subseteq I$ *for all* $\alpha \in I \cap S$;
- *for any* $f : S \cap I \to \kappa$ *s.t.* $f(\alpha) \in g(\alpha) \cap \alpha$ *for all* $\alpha \in S \cap I$, *there is* $\xi^* < \kappa$ *s.t.* $f^{-1}{}''\{\xi^*\}$ *is stationary in* $sup(I)$.

- $E^\kappa_\lambda = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$.

**Facts 7**

*(1) FRP follows from (a weakening of) Axiom R.*
*(2) FRP is consistent with CH (under certain large cardinal axiom)*
*(3) FRP is preserved under c.c.c. generic extensions.*

## countably tight meta Lindelöf spaces

- A space $X$ is **countably tight** if, for any $U \subseteq X$ and $x \in \overline{U}$ there is $U' \in [U]^{\aleph_0}$ s.t. $x \in \overline{U'}$.
- A space $X$ is **meta-Lindelöf** if every open cover of $X$ has a point countable refinement which is also an open cover.

# countably tight meta Lindelöf spaces



- A space $X$ is **countably tight** if, for any $U \subseteq X$ and $x \in \overline{U}$ there is $U' \in [U]^{\aleph_0}$ s.t. $x \in \overline{U'}$.

- A space $X$ is **meta-Lindelöf** if every open cover of $X$ has a point countable refinement which is also an open cover.

## Left-separated spaces



- A space $X$ is *left-separated* if there is a well-ordering $<$ of $X$ s.t. each initial segment of $X$ w.r.t. $<$ is a closed subset of $X$.

## Collectionwise Hausdorff spaces



- A space $X$ is **of local density** $\kappa$ if for very $p \in X$ there is $Y \in [X]^{\leq\kappa}$ s.t. $p \in int(\overline{Y})$.

- A space $X$ is **collectionwise Hausdorff** if any closed discrete subset $D$ of $X$ can be simultaneously separated by disjoint open sets, i.e., if, for any closed and discrete $D \subseteq X$, there is a family $\mathcal{U}$ of pairwise disjoint open sets such that, for all $d \in D$, there is $U \in \mathcal{U}$ with $D \cap U = \{d\}$.

# Collectionwise Hausdorff spaces



- A space $X$ is **of local density** $\kappa$ if for very $p \in X$ there is $Y \in [X]^{\leq\kappa}$ s.t. $p \in int(\overline{Y})$.

- A space $X$ is **collectionwise Hausdorff** if any closed discrete subset $D$ of $X$ can be simultaneously separated by disjoint open sets, i.e., if, for any closed and discrete $D \subseteq X$, there is a family $\mathcal{U}$ of pairwise disjoint open sets such that, for all $d \in D$, there is $U \in \mathcal{U}$ with $D \cap U = \{d\}$.

# Openly generated Boolean algebras



- A Boolean algebra $B$ is said to be **openly generated** iff $\{A \in [B]^{\aleph_0} : A \leq_{\rm rc} B\}$ contains a club set ($\subseteq [B]^{\aleph_0}$)
- $\mathbf{A} \leq_{\rm rc} \mathbf{B}$ $\Leftrightarrow$ $A$ is a relatively complete subalgebra of $B$ $\Leftrightarrow$ $A$ is a subalgebra of $B$ and $\forall b \in B$ (the ideal $A \restriction b$ is generated by a single element (lower projection of $b$)).

**Theorem 8 (S.F., Heindorf, Shapiro, 1994)**

*For a Boolean algebra $B$, the following are equivalent:*

(1) *$B$ is openly generated;*

(2) *$\Vdash_{\mathbb{P}}$ "$B$ is projective" for any $\sigma$-closed $\mathbb{P}$ forcing $|B| = \aleph_1$;*

(3) *$B$ has Freese-Nation property. I.e., there is a mapping (Freese-Nation mapping (or FN-mapping)) $f : B \to [B]^{<\aleph_0}$ s.t. $\forall a, b \in B\, (a \leq b \to \exists c \in f(a) \cap f(b)\ (a \leq c \leq b))$.*

# Openly generated Boolean algebras



- A Boolean algebra $B$ is said to be **openly generated** iff $\{A \in [B]^{\aleph_0} : A \leq_{\rm rc} B\}$ contains a club set ($\subseteq [B]^{\aleph_0}$)

- **$A \leq_{\rm rc}$ B** $\Leftrightarrow$ $A$ is a relatively complete subalgebra of $B$ $\Leftrightarrow$ $A$ is a subalgebra of $B$ and $\forall b \in B$ (the ideal $A \restriction b$ is generated by a single element (lower projection of $b$)).

**Theorem 8 (S.F., Heindorf, Shapiro, 1994)**

*For a Boolean algebra $B$, the following are equivalent:*

(1) *$B$ is openly generated;*

(2) $\Vdash_{\mathbb{P}}$ *"$B$ is projective" for any $\sigma$-closed $\mathbb{P}$ forcing $|B| = \aleph_1$;*

(3) *$B$ has Freese-Nation property. I.e., there is a mapping (Freese-Nation mapping (or FN-mapping)) $f : B \to [B]^{<\aleph_0}$ s.t. $\forall a, b \in B\,(a \leq b \to \exists c \in f(a) \cap f(b)\ (a \leq c \leq b))$.*

# Openly generated Boolean algebras



- A Boolean algebra $B$ is said to be **openly generated** iff $\{A \in [B]^{\aleph_0} : A \leq_{\rm rc} B\}$ contains a club set ($\subseteq [B]^{\aleph_0}$)

- $\mathbf{A} \leq_{\rm rc} \mathbf{B}$ $\Leftrightarrow$ $A$ is a relatively complete subalgebra of $B$ $\Leftrightarrow$ $A$ is a subalgebra of $B$ and $\forall b \in B$ (the ideal $A \restriction b$ is generated by a single element (lower projection of $b$)).

**Theorem 8 (S.F., Heindorf, Shapiro, 1994)**

*For a Boolean algebra $B$, the following are equivalent:*

(1) $B$ *is openly generated;*

(2) $\Vdash_{\mathbb{P}}$ "$B$ *is projective*" *for any $\sigma$-closed $\mathbb{P}$ forcing $|B| = \aleph_1$;*

(3) $B$ *has Freese-Nation property. I.e., there is a mapping (Freese-Nation mapping (or FN-mapping)) $f : B \to [B]^{<\aleph_0}$ s.t. $\forall a, b \in B\,(a \leq b \to \exists c \in f(a) \cap f(b)\ (a \leq c \leq b))$.*

# Openly generated Boolean algebras



- A Boolean algebra $B$ is said to be **openly generated** iff $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club set ($\subseteq [B]^{\aleph_0}$)

- $\mathbf{A} \leq_{\mathrm{rc}} \mathbf{B}$ $\Leftrightarrow$ $A$ is a relatively complete subalgebra of $B$ $\Leftrightarrow$ $A$ is a subalgebra of $B$ and $\forall b \in B$ (the ideal $A \restriction b$ is generated by a single element (lower projection of $b$)).

**Theorem 8 (S.F., Heindorf, Shapiro, 1994)**

*For a Boolean algebra $B$, the following are equivalent:*

(1) $B$ *is openly generated;*

(2) $\Vdash_{\mathbb{P}}$ "$B$ *is projective*" *for any* $\sigma$*-closed* $\mathbb{P}$ *forcing* $|B| = \aleph_1$*;*

(3) $B$ *has Freese-Nation property. I.e., there is a mapping (Freese-Nation mapping (or FN-mapping))* $f : B \to [B]^{<\aleph_0}$ *s.t.* $\forall a, b \in B\,(a \leq b \to \exists c \in f(a) \cap f(b)\ (a \leq c \leq b))$.

# Openly generated Boolean algebras



- A Boolean algebra $B$ is said to be **openly generated** iff $\{A \in [B]^{\aleph_0} : A \leq_{\rm rc} B\}$ contains a club set ($\subseteq [B]^{\aleph_0}$)

- $\mathbf{A} \leq_{\rm rc} \mathbf{B}$ $\Leftrightarrow$ $A$ is a relatively complete subalgebra of $B$ $\Leftrightarrow$ $A$ is a subalgebra of $B$ and $\forall b \in B$ (the ideal $A \restriction b$ is generated by a single element (lower projection of $b$)).

**Theorem 8 (S.F., Heindorf, Shapiro, 1994)**

*For a Boolean algebra $B$, the following are equivalent:*

(1) *$B$ is openly generated;*

(2) $\Vdash_{\mathbb{P}}$ *"$B$ is projective" for any $\sigma$-closed $\mathbb{P}$ forcing $|B| = \aleph_1$;*

(3) *$B$ has Freese-Nation property. I.e., there is a mapping (Freese-Nation mapping (or FN-mapping)) $f : B \to [B]^{<\aleph_0}$ s.t. $\forall a, b \in B\ (a \leq b \to \exists c \in f(a) \cap f(b)\ (a \leq c \leq b))$.*

# Openly generated Boolean algebras



- A Boolean algebra $B$ is said to be **openly generated** iff $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club set ($\subseteq [B]^{\aleph_0}$)

- $\mathbf{A} \leq_{\mathrm{rc}} \mathbf{B}$ $\Leftrightarrow$ $A$ is a relatively complete subalgebra of $B$ $\Leftrightarrow$ $A$ is a subalgebra of $B$ and $\forall b \in B$ (the ideal $A \restriction b$ is generated by a single element (lower projection of $b$)).

**Theorem 8 (S.F., Heindorf, Shapiro, 1994)**

*For a Boolean algebra $B$, the following are equivalent:*

(1) *$B$ is openly generated;*

(2) $\Vdash_{\mathbb{P}}$ "*$B$ is projective*" *for any $\sigma$-closed $\mathbb{P}$ forcing $|B| = \aleph_1$;*

(3) *$B$ has Freese-Nation property. I.e., there is a mapping (Freese-Nation mapping (or FN-mapping)) $f : B \rightarrow [B]^{<\aleph_0}$ s.t. $\forall a, b \in B\, (a \leq b \rightarrow \exists c \in f(a) \cap f(b)\ (a \leq c \leq b))$.*