Reflection number of Rado Conjecture and Fodor-type reflection

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Reflection numbers

- $\mathfrak{Refl}_{\mathsf{RC}}$ = the minimal κ s.t., for any tree T, if every subset of size $< \kappa$ is special then T is also special

  = the minimal κ s.t., for any linear ordering L and any family $\mathcal{A}$ of invervals in L, if any subgraph $\langle \mathcal{B}, I_{\mathcal{B}} \rangle$ of intersection graph $\langle \mathcal{A}, I_{\mathcal{A}} \rangle$ of size $< \kappa$ is of countable chromatic number, then $\langle \mathcal{A}, I_{\mathcal{A}} \rangle$ also is of countable chromatic number (S.Todorčević).

- A subset T' of a tree T is **special** if T' is a union of counrably many subsets T'_n, $n \in \omega$ s.t. elements of each T'_n are pairwise incomparable.

- For x, $y \in \mathcal{A}$, $x \; \mathbf{I}_{\mathcal{A}} \; y$ if and only if $x \neq y$ and $x \cap y \neq \emptyset$.

$\mathfrak{Refl}_{\mathsf{RC}} = \aleph_2 \;\Leftrightarrow\;$ Rado conjecture (RC).

Results in this talk are going to be included in a joint paper in preparation with Hiroshi Sakai, Victor Torres and Toshimichi Usuba. The main result of the talk is obtained by Usuba.

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- $\mathfrak{Refl}_{\mathsf{FRP}}$ = the minimal κ s.t., for any regular $\lambda \geq \kappa$, stationary $E \subseteq E^{\lambda}_{\omega}$ and for any ladder system $g : E \to [\lambda]^{\omega}$, there is $\alpha \in E^{\lambda}_{\geq\omega_1} \cap E^{\lambda}_{<\kappa}$ s.t. $\{x \in [\alpha]^{\aleph_0} : \sup(x) \in E, g(\sup(x)) \subseteq x\}$ is stationary in $[\alpha]^{\aleph_0}$.

$\mathfrak{Refl}_{\mathsf{RC}} = \aleph_2 \;\Leftrightarrow\;$ Rado conjecture (RC).

$\mathfrak{Refl}_{\mathsf{FRP}} = \aleph_2 \;\Leftrightarrow\;$ Fodor-type Reflection Principle (FRP).

- RC and FRP are both consistent with ZFC (under some large cardinal $\lessapprox$ a strongly compact card.).

Reflection numbers (2/3)

- $\mathfrak{Refl}_{chr}$ = the minimal κ s.t., for any graph G, if $chr(H) \leq \omega$ for all $H \in [G]^{<\kappa}$ then $chr(G) \leq \omega$.
- $\mathfrak{Refl}_{col}$ = the minimal κ s.t., for any graph G, if $col(H) \leq \omega$ for all $H \in [G]^{<\kappa}$ then $col(G) \leq \omega$.
  - For a graph $G = \langle G, E \rangle$, $col(\mathbf{G})$ = the minimal cardinal κ s.t. there is a well-ordering $\sqsubset$ of G with the property that $|\{y \in G : y \sqsubset x, x \; E \; y\}| < \kappa$ for all $x \in G$.
- $\mathfrak{Refl}_{RC} \leq \mathfrak{Refl}_{chr}$ (by Todorčević's characteriziation of $\mathfrak{Refl}_{RC}$).
- $\beth_\omega \leq \mathfrak{Refl}_{chr} \leq$ strongly compact card. (Erdős and Hajnal 1968?).
- $\aleph_1 < \mathfrak{Refl}_{col} \leq \mathfrak{Refl}_{FRP}$ (S.F. and H.Sakai).
- $\mathfrak{Refl}_{col} = \aleph_2 \Leftrightarrow \mathfrak{Refl}_{FRP} = \aleph_2$ (S.F., H.Sakai, L.Soukup and T.Usuba).
- $\mathfrak{Refl}_{col} = \infty$ is possible (this holds e.g. under $V = L$).

Reflection numbers (3/3)

Theorem. (to appear in a paper by S.F., H. Sakai, V. Torres and T. Usuba)
$\mathfrak{Refl}_{\mathsf{FRP}} \leq \mathfrak{Refl}_{\mathsf{RC}}$.

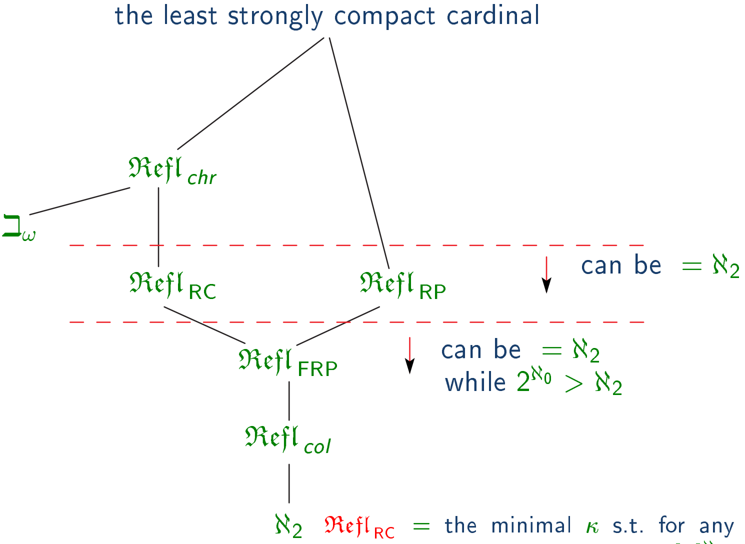
Corollary. (1) Rado Conjecture implies Fodor-type Reflection Principle.

(2) $\mathfrak{Refl}_{col} \leq \mathfrak{Refl}_{chr}$.

Questions.

- ▷ $\mathfrak{Refl}_{col} = \mathfrak{Refl}_{\mathsf{FRP}}$?
- ▷ $\mathfrak{Refl}_{col} \leq \mathfrak{Refl}_{list-chr} \leq \mathfrak{Refl}_{chr}$?
- ▷ Is it consistent that $\mathfrak{Refl}_{\mathsf{FRP}} < \mathfrak{Refl}_{\mathsf{RC}} = \infty$?

Reflection numbers



$\mathfrak{Refl}_{\mathrm{RC}}$ = the minimal κ s.t. for any regular cardinal λ and stationary $S \subseteq [\lambda]^{\aleph_0}$ there is $\alpha \in E^{\lambda}_{\geq\omega_1} \cap E^{\lambda}_{<\kappa}$ with the property that $S \cap [\alpha]^{\aleph_0}$ is stationary in $[\alpha]^{\aleph_0}$.

Proof of the Theorem

Theorem. (to appear in a paper by S.F., H. Sakai, V. Torres and T. Usuba)
$\mathfrak{Refl}_{\mathsf{FRP}} \leq \mathfrak{Refl}_{\mathsf{RC}}$.

Proof. Suppose that κ is a regular cardinal $< \mathfrak{Refl}_{\mathsf{FRP}}$. It is enough to show that $\kappa < \mathfrak{Refl}_{\mathsf{RC}}$ — note that $\mathfrak{Refl}_{\mathsf{FRP}}$ cannot be a successor of a singular cardinal by definition.

- By the assumption, there is a regular λ, stationary $E \subseteq E^{\lambda}_{\omega}$ and a ladder system $g : E \to [\lambda]^{\aleph_0}$ s.t.

$$S = \{x \in [\lambda]^{\aleph_0} \ : \ \sup(x) \notin x,\ g(\sup(x)) \subseteq x\}$$

is stationary (this is always the case) but

$$S_{\alpha} = \{x \in [\alpha]^{\aleph_0} \ : \ \sup(x) \notin x,\ g(\sup(x)) \subseteq x\}$$

for all $\alpha \in E^{\lambda}_{\geq\omega_1} \cap E^{\lambda}_{<\kappa}$ is non-stationary.
- For $x, y \in S$, let $x \prec y \ :\Leftrightarrow\ x \subseteq y$ and $\sup(x) < \sup(y)$.

Proof of the Theorem

$$S = \{x \in [\lambda]^{\aleph_0} \ : \ \sup(x) \notin x,\ g(\sup(x)) \subseteq x\}$$

- For $x, y \in S$, let $x \prec y :\Leftrightarrow x \subseteq y$ and $\sup(x) < \sup(y)$.
- Let **T** be the set of all continuously $\prec$-increasing sequence $t = \langle x_\alpha \ : \ \alpha < \delta \rangle$ in S of length $< \omega_1$ s.t. $\bigcup_{\alpha<\delta} x_\alpha \in S$.
- For $t, t' \in T$ let $t <_{\mathbf{T}} t'$ if t' is an end-extension of t.

- $\mathcal{T} = \langle T, <_T \rangle$ above witnesses $\kappa < \mathfrak{Refl}_{\mathrm{RC}}$:

> Claim.
> (1) $\mathcal{T}$ is not special.
> (2) For every $X \in [S]^{<\kappa}$, $\mathcal{T}^X = \{t \in T \ : \ \bigcup t \subseteq X\}$ is special.

(1): Since $\mathcal{T}$ is a Baire tree, it is not special.

(2): By induction on $otp(X)$.

$\square$

Vielen Dank für die Aufmerksamkeit

Děkuji vám za pozornost !