

# Reflection number of Rado Conjecture and Fodor-type reflection

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This presentation is typeset by p<sup>A</sup>T<sub>E</sub>X with beamer class.

- ▶  $\mathfrak{Refl}_{RC} =$  the minimal  $\kappa$  s.t., for any tree  $T$ , if every subset of size  $< \kappa$  is special then  $T$  is also special

= the minimal  $\kappa$  s.t., for any linear ordering  $L$  and any family  $\mathcal{A}$  of intervals in  $L$ , if any subgraph  $\langle \mathcal{B}, I_{\mathcal{B}} \rangle$  of intersection graph  $\langle \mathcal{A}, I_{\mathcal{A}} \rangle$  of size  $< \kappa$  is of countable chromatic number, then  $\langle \mathcal{A}, I_{\mathcal{A}} \rangle$  also is of countable chromatic number (S.Todorčević).

- ▶ A subset  $T'$  of a tree  $T$  is **special** if  $T'$  is a union of countably many subsets  $T'_n$ ,  $n \in \omega$  s.t. elements of each  $T'_n$  are pairwise incomparable.
- ▶ For  $x, y \in \mathcal{A}$ ,  $x \perp_{\mathcal{A}} y$  if and only if  $x \neq y$  and  $x \cap y = \emptyset$ .

$\mathfrak{Refl}_{RC} = \aleph_2 \Leftrightarrow$  Rado conjecture (RC).

Results in this talk are going to be included in a joint paper in preparation with Hiroshi Sakai, Victor Torres and Toshimichi Usuba. The main result of the talk is obtained by Usuba.

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- ▶  $\mathfrak{Refl}_{FRP} =$  the minimal  $\kappa$  s.t., for any regular  $\lambda \geq \kappa$ , stationary  $E \subseteq E_{\omega}^{\lambda}$  and for any ladder system  $g : E \rightarrow [\lambda]^{\omega}$ , there is  $\alpha \in E_{\geq \omega_1}^{\lambda} \cap E_{< \kappa}^{\lambda}$  s.t.  $\{x \in [\alpha]^{\aleph_0} : \text{sup}(x) \in E, g(\text{sup}(x)) \subseteq x\}$  is stationary in  $[\alpha]^{\aleph_0}$ .

$\mathfrak{Refl}_{RC} = \aleph_2 \Leftrightarrow$  Rado conjecture (RC).

$\mathfrak{Refl}_{FRP} = \aleph_2 \Leftrightarrow$  Fodor-type Reflection Principle (FRP).

- ▷ RC and FRP are both consistent with ZFC (under some large cardinal  $\lesssim$  a strongly compact card.).

## Reflection numbers (2/3)

Reflection number of RC (4/9)

- ▶  $\mathfrak{Refl}_{chr}$  = the minimal  $\kappa$  s.t., for any graph  $G$ , if  $chr(H) \leq \omega$  for all  $H \in [G]^{<\kappa}$  then  $chr(G) \leq \omega$ .
- ▶  $\mathfrak{Refl}_{col}$  = the minimal  $\kappa$  s.t., for any graph  $G$ , if  $col(H) \leq \omega$  for all  $H \in [G]^{<\kappa}$  then  $col(G) \leq \omega$ .
- ▷ For a graph  $G = \langle G, E \rangle$ ,  
 $col(G)$  = the minimal cardinal  $\kappa$  s.t. there is a well-ordering  $\sqsubset$  of  $G$  with the property that  $|\{y \in G : y \sqsubset x, x E y\}| < \kappa$  for all  $x \in G$ .
- ▶  $\mathfrak{Refl}_{RC} \leq \mathfrak{Refl}_{chr}$  (by Todorčević's characterization of  $\mathfrak{Refl}_{RC}$ ).
- ▶  $\beth_\omega \leq \mathfrak{Refl}_{chr} \leq$  strongly compact card. (Erdős and Hajnal 1968?).
- ▶  $\aleph_1 < \mathfrak{Refl}_{col} \leq \mathfrak{Refl}_{FRP}$  (S.F. and H.Sakai).
- ▶  $\mathfrak{Refl}_{col} = \aleph_2 \Leftrightarrow \mathfrak{Refl}_{FRP} = \aleph_2$  (S.F., H.Sakai, L.Soukup and T.Usuba).
- ▶  $\mathfrak{Refl}_{col} = \infty$  is possible (this holds e.g. under  $V = L$ ).

**Theorem.** (to appear in a paper by S.F., H. Sakai, V. Torres and T. Usuba)

$$\mathfrak{Refl}_{\text{FRP}} \leq \mathfrak{Refl}_{\text{RC}}.$$

**Corollary.** (1) Rado Conjecture implies Fodor-type Reflection Principle.

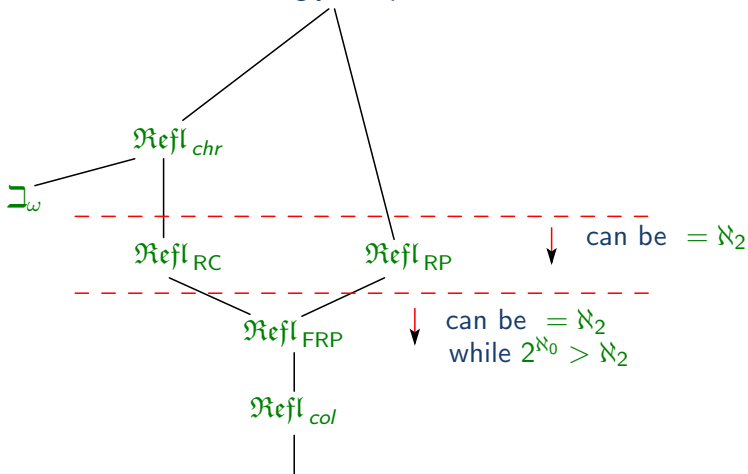
$$(2) \mathfrak{Refl}_{\text{col}} \leq \mathfrak{Refl}_{\text{chr}}.$$

### Questions.

- ▷  $\mathfrak{Refl}_{\text{col}} = \mathfrak{Refl}_{\text{FRP}}$  ?
- ▷  $\mathfrak{Refl}_{\text{col}} \leq \mathfrak{Refl}_{\text{list-chr}} \leq \mathfrak{Refl}_{\text{chr}}$  ?
- ▷ Is it consistent that  $\mathfrak{Refl}_{\text{FRP}} < \mathfrak{Refl}_{\text{RC}} = \infty$  ?

# Reflection numbers

the least strongly compact cardinal



$\aleph_2$   $\text{Refl}_{RC}$  = the minimal  $\kappa$  s.t. for any regular cardinal  $\lambda$  and stationary  $S \subseteq [\lambda]^{\aleph_0}$  there is  $\alpha \in E_{\geq \omega_1}^\lambda \cap E_{< \kappa}^\lambda$  with the property that  $S \cap [\alpha]^{\aleph_0}$  is stationary in  $[\alpha]^{\aleph_0}$ .

**Theorem.** (to appear in a paper by S.F., H. Sakai, V. Torres and T. Usuba)

$$\mathfrak{Refl}_{\text{FRP}} \leq \mathfrak{Refl}_{\text{RC}}.$$

Proof. Suppose that  $\kappa$  is a regular cardinal  $< \mathfrak{Refl}_{\text{FRP}}$ . It is enough to show that  $\kappa < \mathfrak{Refl}_{\text{RC}}$  — note that  $\mathfrak{Refl}_{\text{FRP}}$  cannot be a successor of a singular cardinal by definition.

- By the assumption, there is a regular  $\lambda$ , stationary  $E \subseteq E_\omega^\lambda$  and a ladder system  $g : E \rightarrow [\lambda]^{\aleph_0}$  s.t.

$$S = \{x \in [\lambda]^{\aleph_0} : \sup(x) \notin x, g(\sup(x)) \subseteq x\}$$

is stationary (this is always the case) but

$$S_\alpha = \{x \in [\alpha]^{\aleph_0} : \sup(x) \notin x, g(\sup(x)) \subseteq x\}$$

for all  $\alpha \in E_{\geq \omega_1}^\lambda \cap E_{< \kappa}^\lambda$  is non-stationary.

- For  $x, y \in S$ , let  $x \prec y : \Leftrightarrow x \subseteq y$  and  $\sup(x) < \sup(y)$ .

$$S = \{x \in [\lambda]^{\aleph_0} : \sup(x) \notin x, g(\sup(x)) \subseteq x\}$$

- ▶ For  $x, y \in S$ , let  $x \prec y : \Leftrightarrow x \subseteq y$  and  $\sup(x) < \sup(y)$ .
- ▶ Let  $T$  be the set of all continuously  $\prec$ -increasing sequence  $t = \langle x_\alpha : \alpha < \delta \rangle$  in  $S$  of length  $< \omega_1$  s.t.  $\bigcup_{\alpha < \delta} x_\alpha \in S$ .
- ▶ For  $t, t' \in T$  let  $t <_T t'$  if  $t'$  is an end-extension of  $t$ .
- ▶  $T = \langle T, <_T \rangle$  above witnesses  $\kappa < \mathfrak{Refl}_{RC}$ :

Claim.

(1)  $T$  is not special.

(2) For every  $X \in [S]^{<\kappa}$ ,  $T^X = \{t \in T : \bigcup t \subseteq X\}$  is special.

(1): Since  $T$  is a Baire tree, it is not special.

(2): By induction on  $otp(X)$ .





Děkuji vám za pozornost !

