Rado's conjecture and reflection principles compatible with MM

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Rado's Conjecture and the Reflection Principle RP

- A tree T is **special** if there are $T_i \subseteq T$, $i \in \omega$ s.t. all of T_i's are pairwise incomparable (antichains) and $T = \bigcup_{i \in \omega} T_i$.

- Rado's Conjecture (RC):

> **RC:** Any tree T is special if and only if all subtrees of T of cardinality $\aleph_1$ are special.

- (S. Todorčević) Rado's Conjecture implies the non-existence of Kurepa trees. In particular RC does not hold under $V = L$. Notes

- (S. Todorčević) If κ is strongly compact and $\mathbb{P} = Col(\omega_1, {<}\kappa)$, then we have $\Vdash_{\mathbb{P}}$ "Rado's Conjecture".

- Reflection Principle (RP):

> **RP:** For any regular cardinal $\kappa > \aleph_1$ and stationary $\mathcal{S} \subseteq [\kappa]^{\aleph_0}$, there is $I \in [\kappa]^{\aleph_1}$ s.t. (1) $\omega_1 \subseteq I$, (2) $\mathrm{cf}(I) = \mathrm{cf}(\sup(I)) = \omega_1$ and (3) $\mathcal{S} \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

RC: Any tree T is special if and only if all subtrees of T of cardinality $\aleph_1$ are special.

RP: For any cardinal regular cardinal $\kappa > \aleph_1$ and stationary $\mathcal{S} \subseteq [\kappa]^{\aleph_0}$, there is $I \in [\kappa]^{\aleph_1}$ s.t. (1) $\omega_1 \subseteq I$, (2) $\mathrm{cf}(I) = \mathrm{cf}\,\sup(I) = \omega_1$ and (3) $\mathcal{S} \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

- (S. Todorčević, ??) If $V = L$ we have $\neg$RC and $\neg$RP.
- (S. Todorčević, Foreman-Magidor-Shelah (?))
  If κ is supercompact and $\mathbb{P} = Col(\omega_1, {<}\kappa)$, then we have

  $$\Vdash_{\mathbb{P}} \text{“RC} \ \wedge \ \text{RP”}.$$

Common consequences of RC and RP

- RC and RP have many common consequences:

  - $2^{\aleph_0} \leq \aleph_2$;
  - $\neg\Box_\kappa$ for all κ;
  - Singular Cardinal Hypothesis;
  - Chang's Conjecture (and hence the non existence of Kurepa trees);
  - Ordinal Stationarity Reflection: For any regular $\kappa > \omega_1$ and stationary $S \subseteq E^\omega_\kappa = \{\alpha \in \kappa : \mathrm{cf}(\alpha) = \omega\}$ there is an $\xi \in E^{\omega_1}_\kappa$ s.t. $S \cap \xi$ is stationary in ξ;
  - $\cdots$ (will be discussed later)

Are RC and RP perhaps the same principle?

Are RC and RP perhaps the same principle? Or at least isn't it so that one of them can be drived from the other?

Neither nor!!

- (S. Todorčević) RC implies the negation of Martin's Axiom for $\aleph_1$ dense sets: RP follows from Martin's Maximum.
- ▷ Hence (under a supercompact cardinal) $\neg$RC + RP **is consistent.** (MM implies the combination!)
- (酒井拓史 (H. Sakai)) (Under a supercompact cardinal) it is consistent that there is a strongly compact cardinal κ s.t., for $\mathbb{P} = \mathit{Col}(\omega_1, {<}\kappa)$, we have $\Vdash_{\mathbb{P}}$ "$\neg$RP".
- ▷ Hence (under a supercompact cardinal) RC + $\neg$RP **is consistent.**

RC implies the negation of $\mathrm{MA}_{\aleph_1}$

Theorem 1. (S. Todorčević) $\mathrm{MA}_{\aleph_1}$ implies the negation of RC.

Proof. Let $T = \{t : t$ is an increasing sequence in $\mathbb{R}$ $(\ell(t) < \omega_1)\}$ with the ordering $t <_T t' \Leftrightarrow t'$ is an endextension of t.

Then

- T is a non-special tree.
- Under $\mathrm{MA}_{\aleph_1}$ all subtrees of T of cardinality $\aleph_1$ are special by a theorem of Baumgartner-Malitz-Reinhard:

  • $(\mathrm{MA}_{\aleph_1})$ all trees of cardinality $\aleph_1$ without uncountable chain are special. $\square$

The reason of the similarity of RC and RP

Theorem 2. (1) (P. Doebler, 2013) RC implies the Semi-stationary Reflection Principle (SSR).

(2) (S.F., H. Sakai, V. Torres and T. Usuba, ∞) RC implies the Fodor-type Reflection Principle (FRP).

Since it is already known that RP implies both SSR and FRP, we obtain the diagram:

$$\begin{tikzcd}
\mathrm{RC} \arrow[dr] \arrow[drr] & & & \mathrm{RP} \arrow[dll] \arrow[dl] \\
& \mathrm{FRP} & \mathrm{SSR} &
\end{tikzcd}$$

Common consequences of SSR and FRP

●: SSR implies the assertion.
●: FRP implies the assertion.

- ▷ $2^{\aleph_0} \leq \aleph_2$ ●
- ▷ $\neg\Box_\kappa$ for all κ ● ●
- ▷ Singular Cardinal Hypothesis ● ●
- ▷ Chang's Conjecture (and hence the non existence of Kurepa trees) ●
- ▷ Ordinal Stationarity Reflection: For any regular $\kappa > \omega_1$ and stationary $S \subseteq E^\omega_\kappa = \{\alpha \in \kappa : \mathrm{cf}(\alpha) = \omega\}$ there is an $\xi \in E^{\omega_1}_\kappa$ s.t. $S \cap \xi$ is stationary in ξ ● ●
- ▷ $\cdots$ (will be discussed later) ●

FRP is a "mathematical" reflection principle

- FRP is known to be equivalent (over ZFC) to many "mathematical" reflection theorems such as:

- Any locally countably compact space X is metrizable if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable.

  Sketch of the equivalence proof

- Any T_1-space with point countable base is left separated if every subspaces of X of cardinality $\leq \aleph_1$ are left separated.

- Any graph G is of countable coloring number if all subgraphs Y of X of cardinality $\leq \aleph_1$ are of countable coloring number.

- ...

Variations of Strong Chang's Conjecture

- For a regular cardinal $\kappa > \aleph_1$

CC$^{\downarrow}(\kappa)$: For any sufficiently large regular θ and an well-ordering $\sqsubset$ on $\mathcal{H}(\theta)$, if $\kappa \in M \prec \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$ is countable then, for any $\alpha < \kappa$, there is a countable $M \prec M^* \prec \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$ s.t., letting $\alpha^* = \inf((\kappa \cap M^* \setminus \sup(\kappa \cap M))$, we have $\alpha^* > \alpha$ and $\mathrm{cf}(\alpha^*) = \omega_1$.

CC$^*(\kappa)$: (P. Doebler) For any sufficiently large regular θ and an well-ordering $\sqsubset$ on $\mathcal{H}(\theta)$, if $\kappa \in M \prec \langle \mathcal{H}(\theta), \in, \sqsubseteq \rangle$ is countable then, for any $a \in [\kappa]^{\aleph_1}$, there is a countable $M \prec M^* \prec \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$ s.t., $M \cap \omega_1 = M^* \cap \omega_1$ and there is a $b \in [\kappa]^{\aleph_1} \cap M^*$ s.t. $a \subseteq b$.

$$\begin{tikzcd}
\mathrm{RC} \arrow[d] \arrow[drr] & & \mathrm{RP} \arrow[dll] \arrow[d] \\
\forall \text{ regular } \kappa > \aleph_1 \ \mathrm{CC}^{\downarrow}(\kappa) \arrow[d] & & \forall \text{ regular } \kappa > \aleph_1 \ \mathrm{CC}^{*}(\kappa) \arrow[d] \\
\mathrm{FRP} & & \mathrm{SSR}
\end{tikzcd}$$

Possible Principle(s) unifying FRP and SSR

- $\mathsf{CC}^{\downarrow}(\kappa)$ and $\mathsf{CC}^{*}(\kappa)$ suggest the following natural generalization of the both of the principles:

> $\mathsf{CC}^{\downarrow\downarrow}(\kappa)$: For any sufficiently large regular θ and a well-ordering $\sqsubset$ on $\mathcal{H}(\theta)$, if $M \prec \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$ is countable with $\kappa \in M$ then, for any $a \in [\kappa]^{\aleph_1}$, there are $b \in [\kappa]^{\aleph_1}$ and countable $M \prec M^* \prec \mathcal{M}$ s.t. $a \subseteq b$, $b \in M^*$ and $b \cap M^* = b \cap M$.

- The principle $\mathsf{CC}^{\downarrow\downarrow}(\kappa)$ for all $\kappa > \aleph_1$ clearly implies both $\mathsf{CC}^{\downarrow}(\kappa)$ and $\mathsf{CC}^{*}(\kappa)$. Unfortunately, this principle is actually a strengthening of RP and hence RC does not imply this principle:

> Theorem 3. (S.F., 薄葉季路 (T. Usuba), 2014) For all κ T.f.a.e:
>
> (a) $\mathsf{CC}^{\downarrow\downarrow}(\kappa)$ for all $\kappa > \aleph_1$.
>
> (b) For all κ and stationary $S \subseteq [\kappa]^{\aleph_0}$, for a sufficiently large regular θ and well ordering $\sqsubset$ on $\mathcal{H}(\theta)$, there is an $M \prec \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$ of cardinality $\aleph_1$, s.t. $S \cap M$ is stationary in $[\kappa \cap M]^{\aleph_0}$.

Possible Principle(s) unifying FRP and SSR(2/2)

$$
\begin{tikzcd}
\mathsf{RC} \arrow[drr] & & & & \mathsf{CC}^{\downarrow\downarrow} \arrow[dll] \\
& & ? \arrow[dl] \arrow[dr] & & \\
& \mathsf{CC}^{\downarrow} \arrow[d] & & \mathsf{CC}^{**} \arrow[d] & \\
& \mathsf{FRP} & & \mathsf{SSR} \arrow[d] & \\
& & & \mathsf{CC} &
\end{tikzcd}
$$

Possible Principle(s) unifying RC and RP under CH

- The model obtained by Levy collapsing (by $\mathbb{P} = Col(\omega_1, {<}\kappa)$) a supercompact cardinal can be seen as quite “canonical”.
- Game Reflection Principle GRP^+ of B. Koenig captures many features of this model. In particular GRP^+ implies RC, RP as well as CH.

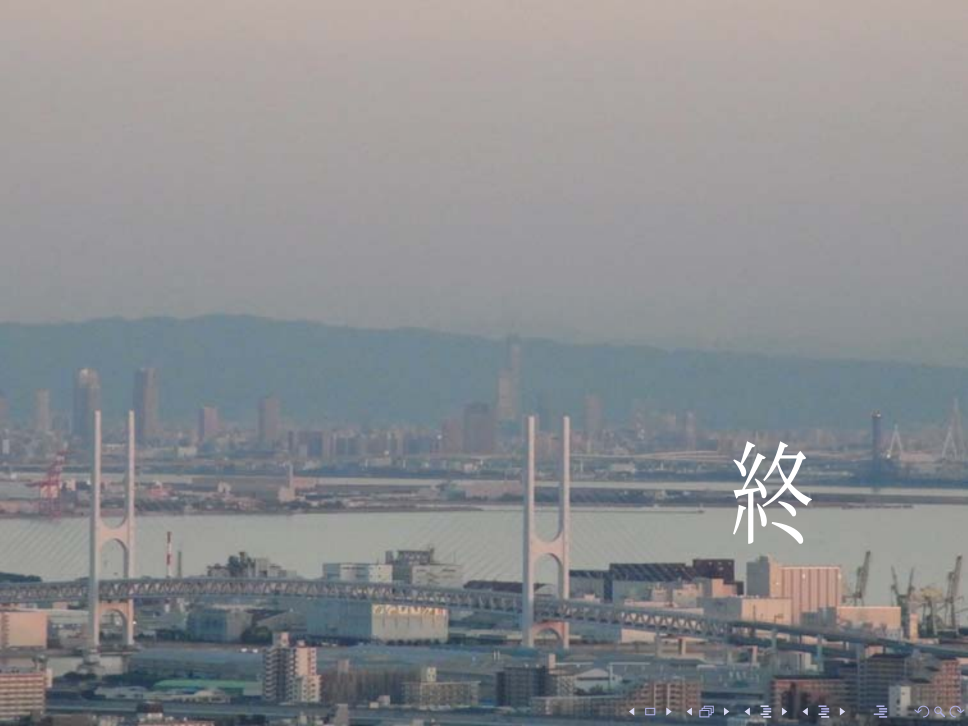
$$
\begin{tikzcd}
& \mathrm{GRP}^+ \arrow[dl] \arrow[drr] & & \\
\mathrm{RC} \arrow[dr] \arrow[drr] & & & \mathrm{RP} \arrow[dll] \arrow[dl] \\
& \mathrm{FRP} & \mathrm{SSR} &
\end{tikzcd}
$$

Possible Principle(s) unifying RC and RP under $\neg$CH

- Under $\neg$CH, "the canonical model $\models$ MM" does not satisfy RC.
- Mitchel's model constructed starting from a supercompact cardinal satisfies both RC and RP under $2^{\aleph_0} = \aleph_2$. This model is however "less canonical".

▷ Is there an axiom which captures a good deal of the characteristics of the Mitchel model?

$$\begin{tikzcd}
 & ? \arrow[dl] \arrow[drr] & & \\
\mathrm{RC} \arrow[dr] \arrow[drr] & & & \mathrm{RP} \arrow[dll] \arrow[dl] \\
 & \mathrm{FRP} & \mathrm{SSR} &
\end{tikzcd}$$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Semi-stationary Reflection Principle

- **Semi-stationary Reflection Principle** (SSR)is the following statement:

  (SSR) For any $W \supseteq \omega_1$ and semi-stationary $X \subseteq [W]^{\aleph_0}$, there is $W' \in [W]^{\aleph_1}$ s.t. $\omega_1 \subseteq W'$ and $X \cap [W']^{\aleph_0}$ is semi-stationary in $[W']^{\aleph_0}$.

  Here, $X \subseteq [W]^{\aleph_0}$ for $W \supseteq \omega_1$ is **semi-stationary** if the set

  $$\{y \in [W]^{\aleph_0} : \exists x \in X\,(x \subseteq y \wedge x \cap \omega_1 = y \cap \omega_1)\}$$

  is stationary in $[W]^{\aleph_0}$.

- SSR is equivalent to the statement of the equivalence:

  Any p.o. $\mathbb{P}$ is starionary ($\subseteq \omega_1$) preserving if and only if it is semiproper (S. Shelah).

Left-separated topological spaces

A topological space X is **left-separated** if there is a well-ordering $<$ of X s.t. all initial segments of X w.r.t. $<$ are closed subsets of X.

$$\cdots](\cdots$$

$$<$$

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Coloring number of a graph

The coloring number $col(G)$ of a graph G is defined by:

- $col(\mathbf{G})$ = the minimal cardinal κ s.t. there is a well-ordering $\sqsubset$ of G with the property that $|\{y \in G : y \sqsubset x, x\, E\, y\}| < \kappa$ for all $x \in G$.
- We have $chr(G) \leq col(G)$.

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Sketch of the equivalence proof

FRP is equivalent to the following assertion (over ZFC):

Any locally countably compact space X is metrizable if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable.

Sketch of the proof: "$\Rightarrow$" is proved by showing first that FRP implies the following reflection statement:

(S.F. I. Juhász, L. Soukup, Z. Szentimiklóssy, T. Usuba, 2010) For every locally separable countably tight topological space X, if all subspaces of X of cardinality $\leq \aleph_1$ are meta-Lindelöf, then X itself is also meta-Lindelöf.

For "$\Leftarrow$" we use the following fact from [S.F., H.Sakai, L.Soukup, T.Usuba, ∞]:

Fact. *If $\neg$FRP holds then there is a regular cardinal λ with $ADS^{-}(\lambda)$: there are stationary $E^* \subseteq E^{\lambda}_{\omega}$ and a ladder system $g^* : E^* \to [\lambda]^{\aleph_0}$ s.t. $g^* \upharpoonright \alpha$ is essentially disjoint for all $\alpha < \lambda$.*

Sketch of the equivalence proof (2/2)

Fact. *If $\neg FRP$ holds then there is a regular cardinal λ with $ADS^{-}(\lambda)$: there are stationary $E^* \subseteq E^\lambda_\omega$ and a ladder system $g^* : E^* \to [\lambda]^{\aleph_0}$ s.t. $g^* \restriction \alpha$ is essentially disjoint for all $\alpha < \lambda$.*

- Let λ, E^*, g^* be as above. We may assume that $g^*(\alpha) \cap E^* = \emptyset$ for all $\alpha \in E^*$.
- Let $X = E^* \cup \bigcup_{\alpha \in E^*} g^*(\alpha)$ and $\mathcal{O}$ be the topology on X generated by
$$\mathcal{B} = \{\{\alpha\} : \alpha \in \textstyle\bigcup_{\alpha \in E^*} g^*(\alpha)\}$$
$$\cup \{g^*(\alpha) \cup \sup\{\alpha\} \setminus x : \alpha \in E^*, x \in [g^*(\alpha)]^{<\aleph_0}\}$$
- Any subspace Y of X of cardinality $< \lambda$ is metrizable: Since $g^*(\alpha)$, $\alpha \in E^* \cap Y$ are essentially disjoint, Y can be partitioned into disjoint metrizable open subspaces.
- X itself is not metrizable since it is not meta-Lindelöf: Consider the open covering $\mathcal{B}$ of X. Fodor's Lemma imples that there is no point countable open refinement.

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$\mathbf{ADS}^-(\lambda)$

For $X \subseteq \lambda$, $g : X \to [\lambda]^{\aleph_0}$ is a **ladder system** if $otp(g(\alpha)) = \omega$ and $g(\alpha)$ is a cofinal subset of α for all $\alpha \in X$.

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$g : X \to \mathcal{P}(Y)$ is **essentially disjoint** if there is $h : X \to [Y]^{<\aleph_0}$ s.t. $g(x) \setminus h(x)$, $x \in X$ are pairwise disjoint.

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Non-existence of Kurepa tree under RC

- An ω_1-tree T is called a Kurepa tree if T has $\geq \aleph_2$ branches.
- RC implies (a strong form of) Chang's Conjecture which implies the non-existence of any Kurepa trees.
- $V = L$ (or more generally the condition $\aleph_2$ is not inaccessible in L) implies that there is a Kurepa tree.
- ▷ Thus, e.g. no generic extension of L satisfies RC.

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