The Mathematical Infinite as a Matter of Method

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Abstract. I address the historical emergence of the mathematical infinite, and how we are to take the infinite in and out of mathematics. The thesis is that *the mathematical infinite in mathematics is a matter of method.*

The infinite, of course, is a large topic. At the outset, one can historically discern two overlapping clusters of concepts: (1) wholeness, completeness, universality, absoluteness. (2) endlessness, boundlessness, indivisibility, continuousness. The first, the *metaphysical infinite*, I shall set aside. It is the second, the *mathematical infinite*, that I will address. Furthermore, I will address mathematical infinite by considering its historical emergence in set theory and how we are to take it in and out of mathematics. Insofar as physics and, more broadly, science deals with the mathematical infinite through mathematical language and techniques, my remarks should be subsuming and consequent.

The main underlying point is that how the mathematical infinite is approached, assimilated, and applied in mathematics is not a matter of "ontological commitment", of coming to terms with whatever that might mean, but rather of epistemological articulation, of coming to terms through knowledge. *The mathematical infinite in mathematics is a matter of method.* How we deal with the specific individual issues involving the infinite turns on the narrative we present about how it fits into methodological mathematical frameworks established and being established.

The first section discusses the mathematical infinite in historical context, and the second, set theory and the emergence of the mathematical infinite. The third section discusses the infinite in and out of mathematics, and how it is to be taken.

§1. The Infinite in Mathematics

What role does the infinite play in modern mathematics? In modern mathematics, infinite sets abound both in the workings of proofs and as subject matter in statements, and so do universal statements, often of $\forall\exists$ "for all there exists" form, which are indicative of direct engagement with the infinite. In many ways the role of the infinite is importantly "second-order" in the sense that Frege regarded number generally, in that the concepts of modern mathematics are understood as having infinite instances over a broad range.

But all this has been the case for just more than a century. Infinite totalities and operations on them only emerged in mathematics in a recent period of algebraic expansion and rigorization of proof. It becomes germane, even crucial, to see how the infinite emerged through interaction with proof to come to see that the infinite in mathematics is a matter of method. If one puts the history of mathematics through the sieve of proof, one sees the emergence of methods drawing in the mathematical infinite, and the mathematical infinite came in at three levels: *the countably infinite*, the infinite of the natural numbers; *the continuum*, the infinite of analysis; and *the empyrean infinite* of higher set theory.

As a thematic entrée into the matter of proof and the countably infinite, we can consider the Pigeonhole Principle:

If n pigeons fly into fewer than n pigeonholes,
then one hole has more than one pigeon.

Taken primordially, this may be considered immediate as part of the meaning of natural number and requires no proof. On the other hand, after its first explicit uses in algebraic number theory in mid-19th Century, it has achieved articulated prominence in modern combinatorics, its consequences considered substantive and at times quite surprisingly so given its immediacy. Rendered as a $\forall\exists$ statement about natural numbers, it however does not have the feel of a basic law of arithmetic but of a "non-constructive" existence assertion, and today it is at the heart of combinatorics, and indeed is the beginning of Ramsey Theory, a field full of non-constructive existence assertions.

So how does one *prove* the Pigeonhole Principle? For 1729 pigeons and 137 pigeonholes one can systematically generate all assignments $\{1, \ldots, 1729\} \longrightarrow \{1, \ldots, 137\}$ and check that there are always at least two pigeons assigned to the same pigeonhole. But we "see" nothing here, nor from any other particular brute force analysis. With the Pigeonhole Principle seen afresh as being at the heart of the articulation of finite cardinality and requiring proof based on prior principles, Richard Dedekind in his celebrated 1888 essay *Was sind und was sollen die Zahlen?* [8, §120] first gave a proof applying the Principle of Induction on n. Today, the Pigeonhole Principle is regarded as a theorem of Peano Arithmetic (PA). In fact, there is a "reverse mathematics" result, in that in the presence of the elementary axioms of PA, the Pigeonhole Principle *implies* the Principle of Induction, and is hence *equivalent* to this central principle. Moreover, the proof complexity of weak forms of the Pigeonhole Principle have been investigated in weak systems of arithmetic.[1]

This raises a notable historical point drawing in the infinite. The Pigeonhole Principle seems to have been first applied in mathematics by Gustav Lejeune Dirichlet in papers of 1842, one to the study of the classical Pell's equation and another to establish a crucial approximation lemma for his well-known Unit Theorem describing the group of units of an algebraic number field.[2] Upon

[1] See for example [18].

[2] See Dirichlet [11, pp.579,636]. The principle in the early days was called the *Schubfachprinzip* ("drawer principle"), though not however by Dirichlet. The second, 1899 edition of Heinrich Weber's *Lehrbuch der Algebra* used the words "in Faecher verteilen" ("to distribute

Gauss's death in 1855 his Göttingen chair went to Dirichlet, but he succumbed to a heart attack a few years later. Dedekind was Gauss's last student, and later colleague and friend of Dirichlet. Already in 1857, Dedekind [6] worked in modular arithmetic with the actually infinite residue classes themselves as unitary objects. His context was in fact $Z[x]$, the polynomials of x with integer coefficients, and so he was entertaining a totality of infinitely many equivalence classes.

The Pigeonhole Principle occurred in Dirichlet's *Vorlesungen über Zahlentheorie* [10], edited and published by Dedekind in 1863. The occurrence is in the second, 1871 edition, in a short Supplement VIII by Dedekind on Pell's equation, and it was in the famous Supplement X that Dedekind laid out his theory of ideals in algebraic number theory, working directly with infinite totalities. In 1872 Dedekind was putting together *Was sind und was sollen die Zahlen?*, and he would be the first to *define* infinite set, with the definition being a set for which *there is* a one-to-one correspondence with a proper subset. This is just the negation of the Pigeonhole Principle. Dedekind in effect had inverted a negative aspect of finite cardinality into a *positive* existence definition of the infinite.

The Pigeonhole Principle brings out a crucial point about method. Its proof by induction is an example of what David Hilbert later called *formal* induction. In so far as the natural numbers do have an antecedent sense, a universal statement $\forall n \varphi(n)$ about them should be correlated with all the informal counterparts to $\varphi(0), \varphi(1), \varphi(2), \ldots$ taken together. *Contra* Poincaré, Hilbert [15] distinguished between *contentual* [*inhaltlich*] induction, the intuitive construction of each integer as numeral,[3] and *formal* induction, by which $\forall n \varphi(n)$ follows immediately from the two statements $\varphi(0)$ and $\forall n(\varphi(n) \rightarrow \varphi(n+1))$ and "through which alone the mathematical variable can begin to play its role in the formal system." In the case of the Pigeonhole Principle, we see the proof by formal induction, but it bears little constructive relation to any particular instance. Be that as it may, the schematic sense of the countably infinite, the infinite of the natural numbers, is carried in modern mathematics by formal induction, a principle used everywhere in combinatorics and computer science to secure statements about the countably infinite. The Pigeonhole Principle is often regarded as surprising in its ability to draw strong conclusions, but one way to explain this is to point to the equivalence with the Principle of Induction. Of course, the Principle of Induction is itself often regarded as surprising in its efficacy, but this can be seen as our reaction to it as method in contrast to the countably infinite taken as primordial. There is no larger mathematical sense to the Axiom of Infinity in set theory other than to provide an extensional counterpart to formal induction, a method of proof. The Cantorian move against the traditional conception of the natural numbers as having no end in the "after" sense is neatly rendered by extensionalizing induction itself in modern set

into boxes") and Edmund Landau's 1927 *Vorlesungen über Zahlentheorie* had "Schubfachschluss".

[3]For Hilbert, the numeral for the integer n consists of n short vertical strokes concatenated together.

theory with the ordinal ω, with "after" recast as "$\in$".

The next level of the mathematical infinite would be the continuum, the infinite of mathematical analysis. Bringing together the two traditional Aristotlean infinities of infinite divisibility and of infinite progression, one can ask: How many points are there on the line? This would seem to be a fundamental, even primordial, question. However, to cast it as a *mathematical* question, underlying concepts would have to be invested with mathematical sense and a way of mathematical thinking provided that makes an answer possible, if not informative. First, the real numbers as representing points on the linear continuum would have to be precisely described. A coherent concept of cardinality and cardinal number would have to be developed for infinite mathematical totalities. Finally, the real numbers would have to be enumerated in such a way so as to accommodate this concept of cardinality. Georg Cantor made all of these moves as part of the seminal advances that have led to modern set theory, eventually drawing in also the third, empyrean infinite of higher set theory. His Continuum Hypothesis would propose a specific, structured resolution about the size of the continuum in terms of his transfinite numbers, a resolution that would become pivotal where set-theoretic approaches to the continuum became prominent in mathematical investigations.

§2. Set Theory and the Emergence of the Infinite

Set theory had its beginnings in the great 19th Century transformation of mathematics, a transformation beginning in analysis. With the function concept having been steadily extended beyond analytic expressions to infinite series, sense for the new functions could only be developed through carefully specified deductive procedures. Proof reemerged in mathematics as an extension of algebraic calculation and became the basis of mathematical knowledge, promoting new abstractions and generalizations. The new articulations to be secured by proof and proof in turn to be based on prior principles, the regress lead in the early 1870s to the appearance of several formulations of the real numbers, of which Cantor's and Dedekind's are the best known. It is at first quite striking that the real numbers as a totality came to be developed so late, but this can be viewed against the background of the larger conceptual shift from intensional to extensional mathematics. Infinite series outstripping sense, it became necessary to adopt an arithmetical view of the continuum given extensionally as a totality of points.

Cantor's formulation of the real numbers appeared in his seminal paper [1] on trigonometric series; proceeding in terms of "fundamental" sequences, he laid the basis for his theorems on sequential convergence. Dedekind [7] formulated the real numbers in terms of his "cuts" to express the completeness of the continuum; deriving the least upper bound principle as a simple consequence, he thereby secured the basic properties of continuous functions. In the use of arbitrary sequences and infinite totalities, both Cantor's and Dedekind's objectifications of the continuum helped set the stage for the subsequent development of that extensional mathematics *par excellence*, set theory. Cantor's formulation was no idle conceptualization, but to the service of specific mathematics, the articulation of his results on uniqueness of trigonometric series involving his derived sets, the first instance of topological closure. Dedekind [7] describes how he came to his formulation much earlier, but also acknowledges Cantor's work. Significantly, both Cantor [1, p.128] and Dedekind [7, III] accommodated the antecedent geometric sense of the continuum by asserting as an "axiom" that each point on the geometric line actually corresponds to a real number as they defined it, a sort of Church's thesis of adequacy for their construals of the continuum. In modern terms, Cantor's reals are equivalence classes according to an equivalence relation which importantly is a congruence relation, a relation that respects the arithmetical structure of the reals. It is through Cantor's formulation that completeness would be articulated for general metric spaces, thereby providing the guidelines for proof in new contexts involving infinite sets.

Set theory was born on that day in December 1873 when Cantor established that *the continuum is not countable*: There is no one-to-one correspondence between the natural numbers $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ and the real numbers $\mathbf{R}$. Like the irrationality of $\sqrt{2}$, the uncountability of the continuum was an impossibility result established via *reductio ad absurdum* that opened up new possibilities. Cantor addressed a specific *problem*, embedded in the mathematics of the time, in his seminal [2] entitled "On a property of the totality of all real algebraic numbers". Dirichlet's algebraic numbers, it will be remembered, are the roots of polynomials with integer coefficients; Cantor established the countability of the algebraic numbers. This was the first substantive correlation of an infinite totality with the natural numbers, and it was the first application of what now goes without saying, that finite words based on a countable alphabet are countable. Cantor then established: *For any (countable) sequence of reals, every interval contains a real not in the sequence.*[4]

By this means Cantor provided a new proof of Joseph Liouville's result [16, 17] that there are transcendental (i.e. real, non-algebraic) numbers, and only afterward did Cantor point out the uncountability of the reals altogether. Accounts of Cantor's existence deduction for transcendental numbers have mostly reserved the order, establishing first the uncountability of real numbers and only then drawing the conclusion from the countability of the algebraic numbers, thus promoting the misconception that his argument is "non-constructive".[5] It depends how one takes a proof, and Cantor's arguments have been implemented as algorithms to generate successive digits of transcendental numbers.[6] The Baire Category Theorem, in its many uses in modern mathematics, has similarly been regarded as non-constructive in its production of examples; however, its proof is an extension of Cantor's proof of the uncountability of the reals, and analogously constructive.

[4]The following is Cantor's argument, in brief: Suppose that s is a sequence of reals and I an interval. Let $a < b$ be the first two reals of s, if any, in I. Then let $a' < b'$ be the first two reals of s, if any, in the open interval (a, b); $a'' < b''$ the first two reals of s, if any, in (a', b'); and so forth. Then however long this process continues, the intersection of the nested intervals must contain a real not in the sequence s.

[5]Indeed, this is where Wittgenstein ([21], part II, 30-41) located what he took to be the problematic aspects of the talk of uncountability.

[6]See Gray [13].

Cantor went on, of course, to develop his concept of cardinality based on one-to-one correspondences. Two totalities have the same cardinality exactly when *there is* a one-to-one correspondence between them. Having made the initial breach with a *negative* result about the lack of a one-to-one correspondence, he established infinite cardinality as a methodologically *positive* concept, as Dedekind had done for infinite set, and investigated the possibilities for *having* one-to-one correspondences. Just as the discovery of the irrational numbers had led to one of the great achievements of Greek mathematics, Eudoxus's theory of geometric proportions, Cantor began his move toward a full-blown mathematical theory of the infinite. By his 1878 *Beitrag* [3] Cantor had come to the focal Continuum Hypothesis—that there is no intervening cardinality between that of the countably infinite and the continuum—and in his 1883 *Grundlagen* [4] Cantor developed the *transfinite numbers* and the key concept of *well-ordering*, in significant part to take a structured approach to infinite cardinality and the Continuum Hypothesis. At the end of the *Grundlagen*, Cantor propounded this basic well-ordering principle: "It is always possible to bring any *well-defined* set into the form of a well-ordered set." Sets are to be well-ordered, and they and their cardinalities are to be gauged via the transfinite numbers of his structured conception of the infinite.

Almost two decades after his [2] Cantor in a short 1891 note [5] gave his now celebrated diagonal argument, establishing *Cantor's Theorem*: *For any set X the totality of functions from X into a fixed two-element set has a larger cardinality than X*, i.e. there is no one-to-one correspondence between the two. This result generalized his [2] result that the continuum is not countable, since the totality of functions from $\mathbf{N}$ into a fixed two-element set has the same cardinality as $\mathbf{R}$. In retrospect the diagonal argument can be drawn out from the [5] proof.[7]

Cantor had been shifting his notion of set to a level of abstraction beyond sets of reals and the like, and the casualness of his [5] may reflect an underlying cohesion with his [2]. Whether the new proof is really "different" from the earlier one, through this abstraction Cantor could now dispense with the recursively defined nested sets and limit construction, and he could apply his argument to any set. He had proved for the first time that there is a cardinality larger than that of $\mathbf{R}$ and moreover affirmed "the general theorem, that the powers of well-defined sets have no maximum." Thus, Cantor for the first time entertained

[7] Starting with a sequence s of reals and a half-open interval I_0, instead of successively choosing delimiting *pairs* of reals in the sequence, avoid the members of s *one* at a time: Let I_1 be the left or right half-open subinterval of I_0 demarcated by its midpoint, whichever does not contain the first element of s. Then let I_2 be the left or right half-open subinterval of I_1 demarcated by its midpoint, whichever does not contain the second element of s; and so forth. Again, the nested intersection contains a real not in the sequence s. Abstracting the process in terms of reals in binary expansion, one is just generating the binary digits of the diagonalizing real.

Cantor first gave a proof of the uncountability of the reals in a letter to Dedekind of 7 December 1873 (Ewald [12, pp.845ff]), professing that ". . . only today do I believe myself to have finished with the thing . . .". It is remarkable that in this letter already appears a doubly indexed array of real numbers and a procedure for traversing the array downward and to the right, as in a now common picturing of the diagonal proof.

the third level of the mathematical infinite, the empyrean level beyond the continuum, of higher set theory.

Nowadays it goes without saying that each function from a set X into a two-element set corresponds to a subset of X, so Cantor's Theorem is usually stated as: *For any set X its power set $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$ has a larger cardinality than X.* However, it would be an exaggeration to assert that Cantor at this point was working on power sets; rather, he was expanding the 19th Century concept of *function* by ushering in arbitrary functions. Significantly, Bertrand Russell was stimulated by the diagonal argument to come up his well-known paradox, this having the effect of emphasizing power sets. At the end of [5] Cantor dealt explicitly with "all" functions with a specified domain X and range $\{0, 1\}$; regarded these as being enumerated by one super-function $\phi(x, z)$ with enumerating variable z; and formulated the diagonalizing function $g(x) = 1 - \phi(x, x)$. This argument, even to its notation, would become method, flowing into descriptive set theory, the Gödel Incompleteness Theorem, and recursion theory, the paradigmatic means of transcendence over an established context.

Cantor's seminal work built in what would be an essential tension of methods in set theory, one that is still very much with us today. His Continuum Hypothesis was his proposed answer to the continuum problem: Where is the size of the continuum in the hierarchy of transfinite cardinals? His diagonal argument, with its mediation for totalities of arbitrary functions (or power sets), would have to be incorporated into the emerging theory of transfinite number. But how is this to be coordinated with respect to his basic 1883 principle that sets should come well-ordered?

The first decade of the new century saw Ernst Zermelo make his major advances, at Göttingen with Hilbert, in the development of set theory. In 1904 Zermelo [22] analyzed Cantor's well-ordering principle by reducing it to the Axiom of Choice (AC), the abstract existence assertion that every set X has a choice function, i.e. a function f such that for every non-empty $Y \in X$, $f(Y) \in Y$. Zermelo thereby shifted the notion of set away from Cantor's principle that every well-defined set is well-orderable and replaced that principle by an explicit axiom. His Well-Ordering Theorem showed specifically that a set is well-orderable exactly when its power set has a choice function. How AC brought to the fore issues about the non-constructive existence of functions is well-known, and how AC became increasingly accepted in mathematics has been well-documented.[8] The expansion of mathematics into infinite abstract contexts was navigated with axioms and proofs, and this led to more and more appeals to AC.

In 1908 Zermelo [23] published the first full-fledged axiomatization of set theory, partly to establish set theory as a discipline free of the emerging paradoxes and particularly to put his Well-Ordering theorem on a firm footing. In addition to codifying generative set-theoretic principles, a substantial motive for

[8] See [19].

Zermelo's axiomatizing set theory was to buttress his Well-Ordering Theorem by making explicit its underlying set existence assumptions.[9] Initiating the first major transmutation of the notion of set after Cantor, Zermelo thereby ushered in a new more abstract, prescriptive view of sets as structured solely by membership and governed and generated by axioms, a view that would soon come to dominate. Thus, the pathways to a theorem played a crucial role by stimulating an axiomatization of a field of study and a corresponding transmutation of its underlying notions.

In the tradition of Hilbert's axiomatization of geometry and of number, Zermelo's axiomatization was in the manner of an implicit definition, with axioms providing rules for procedure and generating sets and thereby laying the basis for proofs. Concerning the notion of definition through axioms, Hilbert [14, p.184] had written already in 1899 as follows in connection with his axiomatization of the reals:

> Under the conception described above [the axiomatic method], the doubts which have been raised against the existence of the totality of all real numbers (and against the existence of infinite sets generally) lose all justification, for by the set of real numbers we do not have to imagine, say, the totality of all possible laws according to which the elements of a fundamental sequence can proceed, but rather—as just described—a system of things, whose mutual relations are given by the *finite and closed* system of axioms I-IV [for complete ordered fields], and about which new statements are valid only if one can derive them from the axioms by means of a finite number of logical inferences.

Zermelo's revelation of the Axiom of Choice for the derivation of the Well-Ordering Theorem was just this, the uncovering of an axiom establishing in a "finite number of logical inferences" Cantor's well-ordering principle, and Zermelo's axiomatization set out a "system of things" given by a "system of axioms".

As with Hilbert's later distinction between contentual and formal induction, infinite sets draw their mathematical meaning not through any direct or intuitive engagement but through axioms like the Axiom of Choice and finite proofs. Just as Euclid's axioms for geometry had set out the permissible geometric constructions, the axioms of set theory would set out the specific rules for set generation and manipulation. But unlike the emergence of mathematics from marketplace arithmetic and Greek geometry, infinite sets and transfinite numbers were neither laden nor buttressed with substantial antecedents. Like strangers in a strange land stalwarts developed a familiarity with them guided hand in hand by their axiomatic framework. For Dedekind in *Was sind und was sollen die Zahlen?* it had sufficed to work with sets by merely giving a few definitions and properties, those foreshadowing the axioms of Extensionality, Union, and Infinity. Zermelo provided more rules, the axioms of Separation, Power

[9] Moore [19, pp.155ff] supports this contention using items from Zermelo's *Nachlass*.

Set, and Choice. Simply put, these rules revealed those aspects to be ascribed to possibly infinite sets for their methodological incorporation into emerging mathematics.[10]

The standard axiomatization ZFC was completed by 1930, with the axiom of Replacement brought in through the work of von Neumann and the axiom of Foundation, through axiomatizations of Zermelo and Bernays. The Cantorian transfinite is contextualized by von Neumann's incorporation of his ordinals and Replacement, which underpins transfinite induction and recursion as methods of proof and definition. And Foundation provides the basis for applying transfinite recursion and induction procedures to get results about *all* sets, they all appearing in the cumulative hierarchy. While Foundation set the stage with a recursively presented picture of the universe of sets, Replacement, like Choice, was seem as an essential axiom for infusing the contextualized transfinite with the order already inherent in the finite.

§3. The Infinite In and Out of Mathematics

It is evident that mathematics has been much inspired by problems and conjectures and has progressed autonomously through the communication of proofs and the assimilation of methods. Being socially and historically contingent, mathematics has advanced when individuals could collectively make mathematics out of concepts, whether they involve infinite totalities or not. The commitment to the infinite is thus to what is communicable about it, to the procedures and methods in articulated contexts, to language and argument. Infinite sets are what they do, and their sense is carried in the methods we collectively employ on their behalf.

When considering the infinite as a matter of method in modern mathematics and its relation to a primordial mathematical infinite, there is a deep irony about mathematical objects and their existence. Through the rigor and precision of modern mathematics, mathematical objects achieve a sharp delineation in mathematical practice as founded on proof. The contextual objectification then promotes, perhaps even urges, some larger sense of reification. Or, there is confrontation with some prior-held belief or sense about existence that then promotes a skeptical attitude about what mathematicians do and prove, especially about the infinite. Whether mathematics inadvertently promotes realist attitudes or not, the applicability of mathematics to science should not extend to philosophy if the issues have to do with existence itself, for again, existence in mathematics is contextual and governed by rules and procedures, and metaphysical existence, especially concerning the infinite, does not inform and is not informed by mathematical work.

Mathematicians themselves are prone to move in and out of mathematics in their existential assessments, stimulated by their work and the urge to put a larger stamp of significance to it. We quoted Hilbert above, and he famously ex-

[10] As is well-known, the Axiom of Choice came to be widely used in ongoing mathematics in the methodologically congenial form of Zorn's Lemma. Significantly, Max Zorn entitled the paper [25] in which he presented his lemma, "A remark on method in transfinite algebra".

pressed larger metaphysical views about finitism and generated a program to establish the overall consistency of mathematics. In set theory, Cantor staked out the Absolute, which he associated with the transcendence of God, and applied it in the guise of the class of all ordinal numbers for delineative purposes. Gödel attributed to his conceptual realism about sets and his philosophical standpoint generally his relative consistency results with the constructible universe L. What are we to make of what mathematicians say and their motivations? Despite the directions in which mathematics has been led by individuals' motivations it is crucial to point out, again, that what has been retained and has grown in mathematics is communicable as proofs and results. We should assess the role of the individual, but keep in mind the larger autonomy of mathematics. A delicate but critical point here is that, as with writers and musicians, we should be dispassionate, sometimes even critical, about what mathematicians say about their craft. As with the surface Platonism often espoused by mathematicians, we must distinguish what is *said* from what is *done*. The interplay between philosophical views of individual mathematicians, historically speaking, and the space of philosophical possibility, both in their times and now, is what needs to be explored.

There is one basic standpoint about the infinite which seems to underly others and leads to prolonged debates about the "epistemic". This is the (Kantian) standpoint of human finitude. We are cast into a world which as a whole must be infinite, yet we are evidently finite, even to the number of particles that makes us up. So how can we come to know the infinite in any substantive way? This long-standing attitude is part of a venerable tradition, and to the extent that we move against it, our approach may be viewed as bold and iconoclastic. Even phenomenologically, we see before us mathematicians working coherently and substantially with infinite sets and concepts. The infinite is embedded in mathematics as method; we can assimilate methods; and we use the infinite through method in proofs. Even those mathematicians who would take some sort of metaphysical stance against the actuality of the infinite in mathematics can nevertheless follow and absorb a proof by mathematical induction.

There is a final, large point in this direction. As mathematics has expanded with the incorporation of the infinite, several voices have advocated the restriction of proof procedures and methods. Brouwer and Weyl were early figures and Bishop, a recent one. How are we to take all this? We now have a good grasp of intuitionistic and constructivist approaches to mathematics as various explicit, worked-out systems. We also have a good understanding of hierarchies of infinitistic methods through quantifier complexity, proof theory, reverse mathematics and the like. Commitments to the infinite can be viewed as the assimilation of methods along hierarchies. Be that as it may, an *ecumenical* approach to the infinite is what seems to be called for: There is no metamathematics, in that how we are able to argue about resources and methods is itself mathematical. As restrictive approaches were advocated, they themselves have been brought into the fold of mathematics, the process itself having mathematical content. Proofs about various provabilities are themselves significant proofs. It is interesting to carry out a program to see how far a strictly finitist or predicativist approach to mathematics can go, not to emasculate mathematics or to tout the one true way, but to find new, informative proofs and to gain an insight into the resources at play, particularly with regard to commitments to infinity.

Stepping back, the study of the infinite in mathematics urges us to develop a larger ecumenicism about the role of the infinite. Like the modern ecumenical approach to proofs in all their variety and complexity, proofs *about* resources provide new mathematical insights about the workings of method. Even then, in relation to later "elementary" proofs or formalized proofs in an elementary system of a statement, prior proofs may well retain an irreducible semantic content. In this content aspects of the infinite reside robustly, displaying the autonomy of mathematics as an evolving practice.

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