PCF THEORY AND CARDIAL INVARIANTS OF THE REALS

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We investigate the *spectrum* of certain cardinal invariants and we show that the pcf theory can be applied to prove certain closure properties of these spectrums.

Definition 1. Given an ideal $\mathcal{I} \subset \mathcal{P}(I)$ let

$$\mathrm{spectrum}_{\text{non-add}}(\mathcal{I}) = \big\{\lambda : \ \exists \mathcal{A} \in \big[\mathcal{I}\big]^{\lambda} \ \cup\mathcal{A} \notin \mathcal{I} \wedge \forall \mathcal{A}' \in \big[\mathcal{A}\big]^{<\lambda} \ \cup\mathcal{A}' \in \mathcal{I}\big\}.$$

Let $\mathfrak{B}$ be the σ-ideal generated by the $\leq$-bounded (or $\leq^*$-bounded) subsets of ω^ω.

Theorem 2. *If* $A \subset \mathrm{spectrum}_{\textit{non-add}}(\mathfrak{B}) \cap \textit{Reg}$*,* $|A| < \mathfrak{h}$*,* $|A| < \mathfrak{b}$*,* $|A| < \min A$*, then* $\mathrm{pcf}(A) \subset \mathrm{spectrum}_{\textit{non-add}}(\mathfrak{B})$*.*

Corollary 3. *If* $\{\aleph_n : n_0 \leq n < \omega\} \subset \mathrm{spectrum}_{\textit{non-add}}(\mathfrak{B})$ *for some* $n_0 < \omega$ *then* $\aleph_{\omega+1} \in \mathrm{spectrum}_{\textit{non-add}}(\mathfrak{B})$*.*

Definition 4. Given an ideal $\mathcal{I} \subset \mathcal{P}(I)$ let

$$\mathrm{spectrum}_{\text{cof}}(\mathcal{I}) = \big\{\lambda : \ \text{there is } \mathcal{A} \subset \mathcal{P}(I) \text{ such that } |\mathcal{A}| = \lambda,$$
$$\cup\mathcal{A} = I, \text{ but } I \setminus \cup\mathcal{A}' \notin \mathcal{I} \text{ for each } \mathcal{A}' \in \big[\mathcal{A}\big]^{<\lambda}\big\}.$$

Theorem 5. *If* $A \subset \mathrm{spectrum}_{\textit{cof}}(\mathfrak{B}) \cap \textit{Reg}$*,* $|A| < \mathfrak{b}$*,* $|A| < \min A$*, then* $\mathrm{pcf}(A) \subset \mathrm{spectrum}_{\textit{cof}}(\mathfrak{B})$*.*

Corollary 6. *If* $\{\aleph_n : n_0 \leq n < \omega\} \subset \mathrm{spectrum}_{\textit{cof}}(\mathfrak{B})$ *for some* $n_0 < \omega$ *then* $\aleph_{\omega+1} \in \mathrm{spectrum}_{\textit{cof}}(\mathfrak{B})$*.*

The following theorem shows that corollaries 3 and 6 are sharp in some sense.

Theorem 7. *If* $\mathrm{Con}\big(\mathrm{ZFC} + \max\mathrm{pcf}(\{\aleph_n : 1 \leq n < \omega\}) > \aleph_{\omega+1}\big)$ then $\mathrm{Con}\big(\mathrm{ZFC} + (\mathrm{spectrum}_{\text{cof}}(\langle\omega^\omega, \leq^*\rangle) \cap \mathrm{spectrum}_{\text{non-add}}(\mathfrak{B})\} \cap \aleph_\omega)$ is infinite but $\aleph_{\omega+1} \notin \mathrm{spectrum}_{\text{non-add}}(\mathfrak{B}) \cup \mathrm{spectrum}_{\text{cof}}(\langle\omega^\omega, \leq^*\rangle)\big)$.

The proof heavily uses the following theorem:

Theorem (Hechler)**.** *If* Q *is a* σ*-directed poset then there is a c.c.c poset* P *such that, in* V^P*,* $\langle\omega^\omega, \leq^*\rangle$ *contains a cofinal subset* $\{f_q : q \in Q\}$ *which is order-isomorphic to* Q*:* $f_q \leq^* f_{q'}$ *iff* $q \leq_Q q'$*.*

Problem 1. *Does the statement of Hechler's theorem hold for* $\langle\mathcal{N}, \subset\rangle$ *or for* $\langle\mathcal{M}, \subset\rangle$*?*

Theorem 8. *If* $A \subset \mathrm{spectrum}_{\textit{non-add}}(\mathcal{N}) \cap \textit{Reg}$*,* A *is countable, then* $\mathrm{pcf}(A) \subset \mathrm{spectrum}_{\textit{non-add}}(\mathcal{N})$*.*

Theorem 9. *If the statement of Hechler's theorem hold for* $\langle\mathcal{N}, \subset\rangle$ *then if* $\mathrm{Con}\big(\mathrm{ZFC} + \max\mathrm{pcf}(\{\aleph_n : 1 \leq n < \omega\}) > \aleph_{\omega+1}\big)$ then $\mathrm{Con}\big(\mathrm{ZFC} + \mathrm{spectrum}_{\text{non-add}}(\mathcal{N}) \cap \aleph_\omega$ is infinite but $\aleph_{\omega+1} \notin \mathrm{spectrum}_{\text{non-add}}(\mathcal{N})\big)$.

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