Ex 1 $\underline{cf(|X|) > \omega}$ $p \notin X$

topology on $X \dot\cup \{p\}$ is generated by:

$\{\{a\} : a \in X\} \cup \{X \cup \{p\} \setminus A : A \in [X]^{<|X|}\}$

$X \dot\cup \{p\}$ is not metrizable: $\chi(p, X \cup \{p\}) > \omega$

Any $Y \in [X \dot\cup \{p\}]^{<|X|}$ is discrete.

Discrete top on Z is metrizable

$d(a,b) = 1$ for all $a \neq b$. $\boxtimes$

<u>Lemma?</u> ω_1 with order topology is a counter example for HH with $< \aleph_1$ in place of $< \aleph_2$

ω_1 is not metrizable:

$\mathrm{Lim}(\omega_1) \cap \omega_1$ is closed but it is not G_δ

<u>Claim</u> If $\mathrm{Lim}(\omega_1) \cap \omega_1 \subseteq O \underset{\text{open}}{\subseteq} \omega_1$ then there is $\alpha < \omega_1$ s.t. $(\alpha, \omega_1) \subseteq O$

$\vdash$ For each $\xi \in \mathrm{Lim}(\omega_1) \cap \omega_1$ let $f(\xi) < \xi < g(\xi)$ be s.t. $(f(\xi), g(\xi)) \subseteq O$ By Fodor there is a stat. $S \subseteq \mathrm{Lim}(\omega_1) \cap \omega_1$ and α^* s.t.

$f(\xi) = \alpha^*$ for all $\xi \in S$

$(\alpha^*, \omega_1) \subseteq O$ $\dashv$

If $\mathrm{Lim}(\omega_1) \cap \omega_1 \subseteq O_n$ $n \in \omega$

$\alpha_n < \omega_1$ is s.t. $(\alpha_n, \omega_1) \subseteq O_n$ then

$\bigcap O_n \supseteq (\underbrace{\sup \alpha_n}_{<\omega_1}, \omega_1) \implies \bigcap O_n \neq \mathrm{Lim}(\omega_1) \cap \omega_1$.

<u>Claim</u> All $\xi < \omega_1$ is conti. and ord. pres. embedded in $\mathbb{R}$.

Prop 3 $\lambda > \omega_1$ regular

If $S \subseteq E^\lambda_\omega = \{\alpha<\lambda : cf(\alpha)=\omega\}$ and

$\xi \cap S$ is non-stat. for all $\xi<\lambda$ with $cf(\xi)>\omega$

Then S as a subspace of order top λ is non metrizable

but each (*) $S \cap \xi$ $\xi<\lambda$ is metrizable

⊢ Claim $Lim(S) \cap S$ is closed in S but it is not G_δ.

Claim (*)

⊢ By induction on ξ,

Lemma (1) For a top space X $X = X_0 \cup \{x_0\}$

x_0 is isolated and X_0 is metrizable

Then X is also metrizable

(2) If $X = \dot{\bigcup}_{i\in I} X_i$ $X_i \subseteq_{open} X$

each X_i is metrizable then X is also metrizable.

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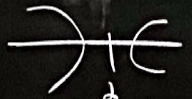
(1) is applied for limit ξ if $cf(\xi)=\omega$



$(..,\alpha)$ α $(\alpha,..)$

$cf(\xi)>\omega$

We find $C \subseteq_{club} \xi$ s.t. $S \cap \xi \cap C = \emptyset$.

For (2): W.l.o.g. d_i is a metric of X_i

s.t. $d_i : (X_i)^2 \longrightarrow [0,1]$

$$\tilde{d}(p,q) = \begin{cases} d_i(p,q) & \text{if } p,q \in X_i \\ 2 & \end{cases}$$

is the metric of X. ⊠

$MA^+(\sigma\text{-closed})$: $\leftarrow MM^+ \leftarrow MM^{++}$

For any σ-closed $\mathbb{P}$, for any $\mathcal{D}$ set of dense subsets of $\mathbb{P}$, $|\mathcal{D}| \leq \aleph_1$ and any $\mathbb{P}$-name $\dot{S}$ s.t.

$\Vdash_{\mathbb{P}} \dot{S} \underset{\text{stat}}{\subseteq} \omega_1$. There is a $\mathcal{D}$-generic filter G on $\mathbb{P}$ s.t.

$\dot{S}[G] \underset{\text{stat}}{\subseteq} \omega_1$

Prop. 7 Suppose $S \underset{\text{stat}}{\subseteq} E$ but $S \cap \xi$ non stat for all $\xi < \lambda$, $cf(\xi) > \omega$

Let $I = \lambda \setminus \mathrm{Lim}(\lambda)$

Let $S \dot{\cup} I$ be the top space with the topology generated by $\{\{\alpha\} : \alpha \in I\} \cup \{\{\xi\} \cup (a_\xi \setminus \beta) ; \beta < \alpha,$

$\xi \in S\}$

For each $\xi \in S$ we define

$a_\xi \subseteq \xi$, $\mathrm{otp}(a_\xi) = \omega$, $\sup(a_\xi) = \xi$

$\cap I$ $\quad$ S

This topol. space will do!

$\Omega_1 \subseteq W \subseteq V$ is a ground of V (W transitive, $\models$ ZFC)

$\iff \exists \mathbb{P} \in W$, $G \in V$ s.t. G is $(W, \mathbb{P})$-generic and

$W[G] = V$

2-huge cardinal

$\downarrow$

ultra huge cardinal

$\downarrow$

super huge

$\downarrow$

huge

$\downarrow$

supercompact

κ is supercompact $\Leftrightarrow$ for $\lambda \geq \kappa$, there is $j, M \subseteq V$ s.t. $j : V \xrightarrow{\preceq}_{\kappa} M$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$ ($\Leftrightarrow$ $j''\lambda \in M$)

κ is $\mathcal{P}$-gen. supercompact $\Leftrightarrow$ for any $\lambda \geq \kappa$ there are $\mathbb{P} \in \mathcal{P}$, $(V, \mathbb{P})$-gen G, $j, M \subseteq V[G]$ s.t.

$\ldots\ldots$ $j''\lambda \in M$

κ is (tightly) $\mathcal{P}$-Laver gen. supercompact $\Leftrightarrow$ for any λ, for any $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $\Vdash_{\mathbb{P}} "\underset{\sim}{\mathbb{Q}} \in \mathcal{P}"$ and for any $(V, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen H then there are $j, M \subseteq V[H]$ $\ldots\ldots$ $\underbrace{j''\lambda \in M}_{\mathbb{P},\ \mathbb{P} * \underset{\sim}{\mathbb{Q}},\ H \in M}$, $(|\mathbb{P} * \underset{\sim}{\mathbb{Q}}| \leq j(\kappa))$

κ is ultrahuge $\Leftrightarrow$

$\ldots\ldots$ $V_{j(\kappa)} \in M$ ${}^{j(\kappa)}M \subseteq M$

κ is (tightly) $\mathcal{P}$-gen. ultrahuge $\Leftrightarrow$

$\ldots\ldots$ $(V_{j(\lambda)})^{V[G]} \in M$ $\mathbb{P}, G \in M$

$\left(j''j(\kappa) \in M\right)$ $|\mathbb{P}| \leq j(\kappa)$

κ is (tightly) $\mathcal{P}$-Laver gen. ultrahuge $\Leftrightarrow$

$\ldots\ldots$ $(V_{j(\lambda)})^{V[H]} \in M$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}, H \in M$

$|\mathbb{P} * \underset{\sim}{\mathbb{Q}}| \leq j(\kappa)$

κ is hyperhuge $\Leftrightarrow$

$\ldots\ldots$ ${}^{j(\lambda)}M \subseteq M$

$\ldots$ $j''j(\lambda) \in M$

$\ldots\ldots$

$j''j(\lambda) \in M$

κ is 2-huge $\Leftrightarrow$ there are $j, M \subseteq V$ s.t. $j : V \xrightarrow{\preceq}_{\kappa} M$ ${}^{j^2(\kappa)}M \subseteq M$

κ is (tightly) super-$C^{(n)}$-$\mathcal{P}$-Laver gen ... $\underbrace{}_{\mathbb{P} \in \mathcal{P}}$

$\Leftrightarrow$ for any $\lambda_0 \geq \kappa$ $\underbrace{}$ there are $\lambda \geq \lambda_0$ and $\underset{\sim}{\mathbb{Q}}$ s.t. $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}$-name $\Vdash_{\mathbb{P}} \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$,

for $(V, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen. H

there are $j, M \subseteq V[H]$ s.t.

$j : V \xrightarrow{\preceq}_{\kappa} M$, $\ldots$, and $V_{j(\lambda)}^{V[H]} \prec_{\Sigma_n} V[H]$.

Proof of Thm 10

(2) Suppose that $\mathbb{X} = (X, \tau)$ is $\mathbb{P}$-[illegible] non-interpolable [illegible] $X = \lambda$

[illegible] $\mathbb{P} \ast (V\mathbb{P})$ [illegible] G [illegible]

Then [illegible] j, $M \subseteq V[G]$ [illegible] $j : V \to M$

$j(\kappa) > \lambda$ $\quad j''\lambda \in M$ [illegible] $|\tau| \leq \lambda$

Then $(j''\lambda, \tau') \in M$ [illegible] character of X is $< \kappa$ $\quad M \models j(X)$ has character $< \kappa$

It follows that $M \models (j''\lambda, \tau')$ is a subspace of $j(X)$

$\tau' = \{ j''U : U \in \tau \}$

$V[G] \models (X, \tau)$ is not interpolable $\quad (X,\tau) \cong (j''\lambda, \tau')$

$\uparrow$ $\mathbb{P}$ preserves non-interpolability of (X,τ) $\qquad \uparrow$ [illegible]

$M \models (j''\lambda, \tau')$ is not interpolable

$M \models$ there is a non-interpolable subspace of $j(X)$ of size $< \lambda < j(\kappa)$

By elementarity

$V \models$ there is a non-interpolable subspace of X of size $< \kappa$

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(3) Replace (X, τ) with $X \cup [\lambda]$, τ_0

$\tau_0 = \tau \cup \{ \{x\} : x \in [\lambda] \setminus X \}$

(c) [illegible] Lemma $\qquad$ [illegible]

Lemma 10A Suppose that $(\mathbb{P}, H(\kappa)) - R_\kappa A$ holds

If X is a non-interpolable space of character $< \kappa$ and of cardinality $< \kappa$ then X is $\mathbb{P}$-indestructibly non-interpolable

Proof If $\mathbb{X} = (X, \tau)$ [illegible] we show that WLOG $|\tau| < \kappa$ If $\Vdash_{\mathbb{P}} (X, \tau)$ is interpolable

when $(X,\tau) \in H(\kappa)$ then there [illegible] $\Sigma(\mathbb{P})$

Then by $(\mathbb{P}, H(\kappa) - R_\kappa A)$ there is a ground W of V s.t. $(X,\tau) \in W$ $\quad W \models (X,\tau)$ is interpolable

But then $V \models (X,\tau)$ is interpolable $\quad \lightning \quad \square$