Theorem A If κ is ccc-gen. supercompact and X is locally countably compact top. sp. s.t. all subspaces of size $< \kappa$ are metrizable then X is metrizable.

Thm 0 The following is consistent and follows from $\mathsf{MA}^+(\sigma\text{-closed})$: If X is locally countably compact and all subspaces of size $< \aleph_2$ are metrizable then X is metrizable.

Remark If κ is ccc-gen. supercompact then κ is weakly inaccessible $\geq 2^{\aleph_0}$ and $= 2^{\aleph_0}$ is consistent. The statement of Thm 0 is independent over $(\mathrm{ccc}, \mathcal{H}(\kappa))$-$\mathsf{RcA}^+$

Theorem D If $(\mathcal{P}, \mathcal{H}(\kappa))$-$\mathsf{RcA}^+$ holds for a class $\mathcal{P}$ of posets s.t. $\mathcal{P}$ is iterable (i.e. $\mathcal{P}$ is closed wrt restriction, forcing equivalence, 2 step iteration and $\mathbb{1} \in \mathcal{P}$) and includes all ccc posets.

Then for any $\mathbb{P} \in \mathcal{P}$ s.t. $\Vdash_{\mathbb{P}} \mathsf{BFA}_{<\kappa}(\mathcal{P})$, Σ_1-formula φ and $a \in \mathcal{H}(\kappa)^V$, $\Vdash_{\mathbb{P}} \varphi(a)$ implies $V \models \varphi(a)$.

$\mathcal{F}$ is point countable if for all $p \in \bigcup \mathcal{F}$ $\{A \in \mathcal{F} : p \in A\}$ is countable. $\mathcal{F}'$ is a refinement of $\mathcal{F}$ if $\bigcup \mathcal{F}' = \bigcup \mathcal{F}$ and for each $A' \in \mathcal{F}'$ there is $A \in \mathcal{F}$ s.t. $A' \subseteq A$. If $\mathcal{U}$ is an open covering of $\underbrace{X}_{\text{a top. sp.}}$ i.e. $\bigcup \mathcal{U} = X$ and each $U \in \mathcal{U}$ is open, then $\mathcal{U}'$ is an open refinement if it is a refinement and each $U \in \mathcal{U}'$ is open.

Lemma 1 For a ccc po $\mathbb{P}$ and a top. space $X = (X, \tau)$ if $\mathcal{U}$ is an open covering of X and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathcal{V}}$ is point countable open refinement of $\check{\mathcal{U}}$". Then there is a point countable open refinement $\underbrace{}_{\mathcal{V}_0}$ of $\mathcal{U}$ (in V).

Proof $\Vdash_{\mathbb{P}} |\underset{\sim}{\mathcal{V}}| = \lambda$ w.l.o.g. (otherwise replace $\mathbb{P}$ with some $\mathbb{P} \restriction p$)

Let $\langle \underset{\sim}{u}_\alpha : \alpha < \lambda \rangle$ be a sequence of $\mathbb{P}$-names s.t.

$\Vdash_{\mathbb{P}}$ "$\langle \underset{\sim}{u}_\alpha : \alpha < \lambda \rangle$ is a injective enumeration of $\underset{\sim}{\mathcal{V}}$ and $\underset{\sim}{f} : \lambda \to \check{\mathcal{U}}$ s.t. $\underset{\sim}{u}_\alpha \subseteq \underset{\sim}{f}(\alpha) \in \check{\mathcal{U}}$"

$\mathcal{V}_0 := \{U_\alpha : \alpha < \lambda\}$ where

$U_\alpha = \{p \in X : p \Vdash_{\mathbb{P}} p \in \underset{\sim}{u}_\alpha$ for some $p \in \mathbb{P}\}$ ... $\boxtimes$

A top. space X is meta-Lindelöf if any open covering of X has a point countable open refinement.

Fact 2 If X is metrizable then X is meta-Lindelöf

(paracompact $\Rightarrow$ meta-Lindelöf)

$\uparrow$ Stone

metrizable

Proposition 3 If κ is ccc-gen. supercompact then for any non-meta Lindelöf space X of character $< \kappa$ there is a subspace Y of X s.t. $|Y| < \kappa$ and Y is non-meta Lindelöf

Proof Suppose wlog that the underlying set of X is λ and $|\tau| = \lambda$. Let $\mathbb{P}$ be a ccc po s.t. for $(V, \mathbb{P})$-generic G there are $j, M \subseteq V[G]$ s.t. $j : V \xrightarrow{\prec}_{\kappa} M$ $j(\kappa) > \lambda$ and $j''\lambda \in M$.

In M let $Y = j(X) \cap j''\lambda$. Then

$V[G] \models Y \cong X$ (we need here the condition X is of character $< \kappa$)

By Lemma 1 $V[G] \models \underset{\cong Y}{X}$ is not meta Lindelöf

It follows that $M \models Y$ is not meta-Lindelöf.

$M \models$ there is a subspace Y of $j(X)$ of cardinality $< j(\kappa)$ which is non meta-Lindelöf

By elementarity

$V \models$ there is a subspace Y of X of cardinality $< \kappa$ ... $\boxtimes$

Theorem 4 (Dow) If X is countably compact and $<\aleph_2$-metrizable (all subspaces of size $<\aleph_2$ are metrizable) then X is metrizable.

<u>Proposition 5</u> (S.F.-Juhász-Soukup-Szentmiklóssy-Usuba)

If X is locally countably compact meta-Lindelöf and $<\aleph_2$-metrizable then X is metrizable

<u>A sketch of a proof:</u> Let $\mathcal{U}$ be an open covering of X s.t. for each $U \in \mathcal{U}$ $\overline{U}$ is countably compact and hence by Dow metrizable. By meta Lindelöfness of X there is a <u>point countable</u> open refinement $\mathcal{V}$ of $\mathcal{U}$ for each $U \in \mathcal{V}$ $\overline{U}$ is metrizable so $\overline{U}$ is countably compact so $\overline{U}$ is <u>separable</u>. There are partition $\mathcal{P}$ of $\mathcal{V}$ s.t. $P \in \mathcal{P}$ is countable and $P, P' \in \mathcal{P}$ $\bigcup P, \bigcup P'$ are disjoint.

$\{\bigcup P : P \in \mathcal{P}\}$ is a partition of X into <u>open sets</u>

clopen sets

each $\bigcup P$ is metrizable as a countable union of metrizable spaces (Urysohn's metrizable).

Thus X is metrizable. $\boxtimes$

<u>Proof of Theorem A</u> Suppose that X is as in the statement of Theorem A. Then X is $<\aleph$ meta-Lindelöf by the assumption and Fact 2. Thus by Proposition 3 X is meta-Lindelöf. By Prop. 5, it follows that X is metrizable

$\boxtimes$

$(\mathcal{P}, H(\kappa))$-RcA$^+$ $\left(\Leftrightarrow \mathrm{MP}(\mathcal{P}, H(\kappa)) \right)$

Recurrence Ax. Maximality Principle

$\Leftrightarrow$ If $\mathbb{P} \in \mathcal{P}$, ψ a Σ_2 formula, $a \in H(\kappa)$ with $\Vdash_{\mathbb{P}} \psi(a)$ then there is a $\mathcal{P}$-ground W of V (i.e. W is a inner model of V and $V = W[G]$ for some $(W, \mathbb{Q})$-generic G, where $W \models \mathbb{Q} \in \mathcal{P}$, $\underbrace{\mathbb{Q} \in W}_{\text{for some}}$, $a \in W$)

$W \models \psi(a)$

Bounded Forcing Axiom

complete Boolean alg.

Let $\mathbb{P}$ be complete Boolean (i.e. $\mathbb{P} = \mathbb{B} \setminus \{0_{\mathbb{B}}\}$)

$\mathrm{BFA}_{<\kappa}(\mathcal{P}) :\Leftrightarrow$ for any $\mathbb{P} \in \mathcal{P}$, for any set $\mathcal{D}$ of maximal antichains $A \in \mathcal{D}$ with $|A| < \kappa$, $|\mathcal{D}| < \kappa$ there is a $\mathcal{D}$-gen. filter in $\mathbb{P}$.

Theorem 8 Suppose that $(\mathcal{P}, H(\kappa))$-RcA [redacted] holds $(\kappa \geq \aleph_2)$. If $\mathbb{P} \in \mathcal{P}$, any Σ_1-formula φ and $a \in H(\kappa)$, if $\Vdash_{\mathbb{P}} \varphi(a)$ then $V \models \varphi(a)$.

Proof Suppose $\mathbb{P}, \varphi, a$ are as above [redacted] and $\Vdash_{\mathbb{P}} \varphi(a)$. [redacted]

By $(\mathcal{P}, H(\kappa))$-RcA, [redacted] there is a [redacted] ground W of V s.t. $W \models \varphi(a)$.

Since φ is Σ_1 it follows that $V \models \varphi(a)$. $\boxtimes$

Theorem 7 (Bagaria Absoluteness Theorem) TFAE (a) $BFA_{<\kappa}(\mathcal{P})$

(b) for any $\mathbb{P} \in \mathcal{P}$, Σ_1 φ and $a \in H(\kappa)$,

$$\Vdash_{\mathbb{P}} \varphi(a) \Rightarrow \varphi(a).$$

$\square$

Cor. 8

$(\mathcal{P}, H(\kappa))$-RcA implies $BFA_{<\kappa}(\mathcal{P})$.