Thm 1 (Bagaria's Absoluteness Theorem)

For any [redacted] κ and a class of posets $\mathcal{P}$ consisting of cB posets, tfae:

(a) $BFA_{<\kappa}(\mathcal{P})$

(b) For any $\mathbb{P}\in\mathcal{P}$, Σ_1-formula φ and $a\in H(\kappa)$

$\Vdash_{\mathbb{P}} \varphi(a) \iff \varphi(a)$ as $H(\mu^+)$, $\mu=|trcl(a)|$

(c) For any $\mathbb{P}\in\mathcal{P}$ and $(V,\mathbb{P})$-gen. G

$\langle H(\kappa)^V,\in\rangle \prec_{\Sigma_1} \langle H(\kappa)^{V[G]},\in\rangle$

Def. $BFA_{<\kappa}(\mathcal{P}) :\iff$ For any $\mathbb{P}\in\mathcal{P}$,

$\mathcal{I}=\{I_\alpha : \alpha<\mu\}$ for $\mu<\kappa$, $\underset{\text{max antichain}}{I_\alpha\subseteq\mathbb{P}}$, $|I_\alpha|<\kappa$

there is a filter G on $\mathbb{P}$ s.t. $G\cap I_\alpha\neq\emptyset$

for any $\alpha<\mu$. (we assume $\mathbb{P}$ is cBa-like (i.e. $\mathbb{P}=\mathbb{B}\setminus\{0_{\mathbb{B}}\}$ for a cBa $\mathbb{B}$).

Proof of Thm 1 (b) $\iff$ (c) is trivial since

for a cardinal κ $H(\kappa)\prec_{\Sigma_1} V$ (provable in ZFC)

[Suppose $a\in H(\kappa)$ then $trcl(a)\subseteq H(\kappa)$. Let $V_\delta\prec_{\Sigma_1}V$

Let $trcl(a)\subseteq M\prec V_\delta$, $|M|<\kappa$

Let N be transitive collapse of M (π) $N\models\varphi(\pi(a))$, $\pi(a)=a$, $N\subseteq H(\kappa)$]

Thm 2 (Ikegami–Tsang + slight extension) For any regular κ

for any class $\mathcal{P}$ of (cB) posets ($\{1\}\in\mathcal{P}$) TFAE

(a) $(\mathcal{P},H(\kappa))_{\Sigma_1}$-$RcA^+$ (maximality principle for Σ_1 wrt $\mathcal{P}$, $H(\kappa)$)

(b) $(\mathcal{P},H(\kappa))_{\Sigma_1}$-$RcA$,

(c) $BFA_{<\kappa}(\mathcal{P})$

Proof: (a) $\Rightarrow$ (b): trivial

(b) $\Rightarrow$ (c): proved last time

(c) $\Rightarrow$ (a): We use Thm 1

Suppose that $\Vdash_{\mathbb{P}}\varphi(a)$

By Bagaria's Thm

It follows that $\varphi(a)$ is potential

So there is a ground of V satisfying $\varphi(a)$. ∎

Def.

$(\mathcal{P}, H(\kappa))_{\Gamma}$-$RcA^+$

$:\iff$ For any $\mathbb{P}\in\mathcal{P}$ and $a\in H(\kappa)$

if φ is a Γ-formula and $\Vdash_{\mathbb{P}}\varphi(a)$

Then there is a $\mathcal{P}$-ground W of V

s.t. $a\in W$ and $W\models\varphi(a)$

Thm 3 For a class of posets $\mathcal{P}$ s.t. $\{1\} \in \mathcal{P}$ and all $\mathbb{P} \in \mathcal{P}$ are stationary preserving ($\mathcal{P}$ is stat. pres.) (⊛)

If $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$-$\mathrm{RcA}^+$

then for any Σ_1-formula φ in $\mathcal{L}_{\in, I_{NS}}$ and $a \in \mathcal{H}(\kappa)$

for any $\mathbb{P} \in \mathcal{P}$: $\Vdash_{\mathbb{P}} \varphi(a) \Leftrightarrow \varphi(a)$

Def. $I_{NS} = \{A \subseteq \omega_1 : A \text{ is non stationary}\}$, $\mathcal{L}_{\in, I_{NS}} = \{\in, I(\cdot)\}$

Lemma 4 (1) If M is transitive (model of enough set theory) $M \supseteq I_{NS}$

$(M, \in, I_{NS}) \models \varphi(a) \Leftrightarrow \varphi(a)$

for Σ_0-formula in $\mathcal{L}_{\in, I_{NS}}$

(2) If $M \subseteq N$ transitive models of enough ZFC with $I_{NS}^M = I_{NS}^N \cap M$ then

For Σ_1-formula φ and $a \in M$ if $M \models \varphi(a)$ then $N \models \varphi(a)$

Lemma 5 If φ is Σ_1 in $\mathcal{L}_{\in, I_{NS}}$ then φ is equivalent to Σ_2 in $\mathcal{L}_{\in}$ (over ZFC)

proof

$$I(x) \Leftrightarrow \underbrace{\exists y (\underbrace{y \subseteq \omega_1 \wedge y \text{ is club} \wedge x \cap y = \emptyset}_{\Sigma_1})}_{\Sigma_1}$$

$\neg I(x)$ is Π_1

Proof of Thm 3 Suppose that φ is Σ_1 in $\mathcal{L}_{\in, I_{NS}}$, $a \in \mathcal{H}(\kappa)$ and $\mathbb{P} \in \mathcal{P}$. If $\varphi(a)$ then by ⊛, Lemma 6 and Lemma 4 we have $\Vdash_{\mathbb{P}} \varphi(a)$

Suppose now $\Vdash_{\mathbb{P}} \varphi(a)$. Then by Lemma 5 and $(\quad)_{\Sigma_2}$-RcA^+ there is a $\mathcal{P}$-ground $W \subseteq V$ s.t. $W \ni a$ and $W \models \varphi(a)$

Since V is a $\mathcal{P}$-generic extension of W we have the assumption of Lemma 4 (2)

Thus we have $V \models \varphi(a)$. ⊠

Lemma 6 If $\mathbb{P}$ is stat preserving then for $(V, \mathbb{P})$-gen G

$$I_{NS}^V = I_{NS}^{V[G]} \cap V.$$

⊠

Thm 7 (Bagaria) For a regular $\kappa > \omega_1$ and a class $\mathcal{P}$ (closed under ...) of posets, TFAE (a) $BFA^{+<\kappa}_{<\kappa}(\mathcal{P})$,

(b) For any $\mathbb{P} \in \mathcal{P}$, Σ_1-formula in $\mathcal{L}_{\varepsilon, I_{NS}}$ and $a \in H(\kappa)$ $\Vdash_{\mathbb{P}} \varphi(a) \Leftrightarrow \varphi(a)$

(c) For $\mathbb{P} \in \mathcal{P}$ and $(V,\mathbb{P})$-generic $\mathbb{G}$

$$\langle H(\kappa)^V, \in, I_{NS}^V \rangle \prec_{\Sigma_1} \langle H(\kappa)^{V[\mathbb{G}]}, \in, I_{NS}^{V[\mathbb{G}]} \rangle$$

Def. $BFA^{+<\kappa}_{<\kappa}(\mathcal{P}) \Leftrightarrow$ For any $\mathbb{P} \in \mathcal{P}$ and any family $\{I_\xi : \xi < \mu\}$ of max. anti-chains in $\mathbb{P}$ with $|I_\xi| < \kappa$, $\mu < \kappa$ and a family $\{\dot{S}_\xi : \xi < \mu\}$ $\mathbb{P}$-names for a stationary subset of ω_1, there is a filter $G \subseteq \mathbb{P}$ s.t. $G \cap I_\xi \neq \emptyset$, $\xi < \mu$ and (b)

(b) $\dot{S}_\xi[G] \subseteq \omega_1$ stat

⊠

Cor For $\mathcal{P}$ with $\{1\} \in \mathcal{P}$ stat preserving

$(\mathcal{P}, H(\kappa))_{\Sigma_2}$-RcA$^+$ implies $BFA^{+<\kappa}_{\kappa}(\mathcal{P})$.

Thm 8 If $(\mathcal{P}, H(\kappa))$-RcA$^+$, $a \in H(\kappa)$, (hold) for all $\mu < \kappa$, φ Σ_2 formula, if $\mathbb{P} \in \mathcal{P}$ forces $BFA_{<\kappa}(\mathcal{P})$. Then $\Vdash_{\mathbb{P}} H(\mu^+) \models \varphi(a)$ then $H(\mu^+) \models \varphi(a)$

(i.e. $H(\mu^+)^V \prec_{\Sigma_2} H(\mu^+)^{V[\mathbb{G}]}$)

$\mathbb{G}$: $(V,\mathbb{P})$-g.

Proof Suppose $\Vdash_{\mathbb{P}} H(\mu^+) \models \varphi(a)$ then by $(\mathcal{P}, H(\kappa))$-RcA$^+$ there is a $\mathbb{Q}$ ground $W \subseteq V$ s.t. $a \in W$ and $W \models BFA_{<\kappa}(\mathcal{P})$, $H(\mu^+) \models \varphi(a)$

$\Updownarrow$

$\exists y\, \psi(a,y)$

$\uparrow$

Π_1

Bagaria's Absoluteness Thm

We apply Thm 1 on this and we obtain

$V \models H(\mu^+) \models \varphi(a)$. ⊠

Thm 9 Assume that $(\mathcal{P}, \mathcal{H}(\kappa))$-RcA$^+$ holds. If $\mathbb{P}$ forces $BPA^{+<\kappa}_{\kappa}(\mathcal{P})$, then, for all $\mu < \kappa$

$$\langle \mathcal{H}(\mu^+)^V, \in, I^V_{NS} \rangle \prec_{\Sigma_2} \langle \mathcal{H}(\mu^+)^{V[\mathbb{G}]}, \in, I^{V[\mathbb{G}]}_{NS} \rangle$$

for $(V, \mathbb{P})$-generic $\mathbb{G}$.

Proof. Similarly to Thm 8 using Thm 7 in place of Thm 1.

⊠

S.F., T.Usuba, On Recurrence Axioms, Theorem 3.1 (p-Lg-RcA-0)

Suppose that κ is tightly $\mathcal{P}$-Laver-gen. ultrahuge for an iterable class $\mathcal{P}$ of posets. Then $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$-RcA$^+$ holds.

S.F., T.Usuba, On Recurrence Axioms, Theorem 4.10 (p-Lg-RcA-5)

Suppose that $\mathcal{P}$ is an iterable class of posets and κ is super-$C^{(\infty)}$-$\mathcal{P}$-Laver-gen. ultrahuge. Then $(\mathcal{P}, \mathcal{H}(\kappa))$-RcA$^+$ holds.