A matrix (matrices in pl.) is a rectangular list of entries

Ex.

$$\begin{array}{cc} & \text{column} \\ \text{rows} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \end{array}$$

1. row

2. sometimes we write $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

1. column

3×3 matrix

$(3,3)$ matrix

$m\times n$ matrix

(m,n) ...

$A, B, C \ldots$

We write

$$\begin{array}{cc} & \qquad\qquad\quad l \\ A = & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ k \cdots & \cdots & a_{kl} & & \vdots \\ a_{m1} & a_{m2} & \cdots & & a_{mn} \end{bmatrix} \end{array}$$

(k,l)-entry of the matrix A

$$= [a_{kl}] = [a_{k,l}]_{\substack{1\le k\le m \\ 1\le l\le n}}$$

or

Addition of matrices

For $m\times n$ matrices A, B with $A=[a_{ij}]$ $B=[b_{ij}]$

we define

$$A+B = C = [c_{ij}] \text{ where } c_{ij} = a_{ij}+b_{ij}$$

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 7 & 1 & 5 \end{bmatrix}$$

Multiplication of matrices

For matrices of A, B of size $l\times m$ and $m\times n$

Example

1. $$\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}} \underset{2\times 4}{\begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix}} = \underset{3\times 4}{\begin{bmatrix} 1\times 1+2\times(-1) & 1\times 0+2\times 1 & 1\times 1+2\times 0 & 1\times 1+2\times 1 \\ 3\times 1+4\times(-1) & \cdots & & \\ & \cdots & & \end{bmatrix}}$$

If $A=[a_{ij}]$ $B=[b_{jk}]$

$$AB = [c_{ik}]_{\substack{1\le i\le l \\ 1\le k\le m}}$$

where

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{im}b_{mk} = \sum_{j=1}^{m} a_{ij}b_{jk}$$

Σ : S

sum

$$\begin{bmatrix}2&3&1\\4&5&-1\end{bmatrix}\begin{bmatrix}1&2\\2&0\\3&5\end{bmatrix}=\begin{bmatrix}11&9\\11&3\end{bmatrix}$$

$$\begin{bmatrix}1&2\\2&0\\3&5\end{bmatrix}\begin{bmatrix}2&3&1\\4&5&-1\end{bmatrix}=\begin{bmatrix}10&13&-1\\4&6&2\\26&34&-2\end{bmatrix}$$

AB is not always equal to BA

A matrix A is called square matrix if A is $m\times m$ matrix for some m

If A is $m\times m$ matrix and B is $m\times n$ matrix

then AB is $m\times n$ matrix

BA

$$\begin{bmatrix}2&3\\1&4\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}11&16\\13&18\end{bmatrix}$$

$\neq$

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}2&3\\1&4\end{bmatrix}=\begin{bmatrix}4&11\\10&21\end{bmatrix}$$

The multiplication of matrices is non-commutative.

Zero matrix and unit matrix

An $n\times m$ matrix $A=[a_{ij}]$ is called zero-matrix if $a_{ij}=0$ for all $1\le i\le n$ and all $1\le j\le m$. A zero matrix of size $m\times n$ is denoted by O or $O_{m\times n}$

capital o

For $m\times n$ matrix A we have $(O=O_{m\times n})$

$A+O=A$

$O+A=A$

$\left[A+O=[a_{ij}]+[0]=[a_{ij}+0]=[a_{ij}]=A\right]$

For $m\times n$ matrices A, B

$A+B=B+A$

$\left[A+B=[a_{ij}+b_{ij}]=[b_{ij}+a_{ij}]=B+A\right]$

An $m\times m$ matrix A is said to be a unit matrix if

$$a_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\\vdots&\vdots&0\\0&\cdots&0\end{bmatrix}$$

$\boxed{\text{cf } 8\times 1=8,\ 1\times 8=8}$

Einheit (German) unit

$$E=\underbrace{\begin{bmatrix}1&&0\\&\ddots&\\0&&1\end{bmatrix}}_{}\Big\}m$$

E_m

(In English I for unit matrix for "identity")

$\boxed{\text{For } m\times m \text{ matrix } A \quad AE=A \quad EA=A}$

Lemma 1.1 $AE = A$ $EA = A$.

$E = [e_{ij}]$

the (i,j) entry of AE

$= \sum_{k=1}^{m} a_{ik} e_{kj}$ $e_{jj} = 1$, $e_{kj} = 0$ $k \neq j$

$[a_{ij}][e_{ij}]$

$= a_{ij} \times 1$

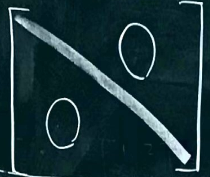
$e_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$=$ (i,j) entry of A

Thus $AE = A$

Similarly $EA = A$

An $m \times m$ matrix A is called diagonal matrix ($A = [a_{ij}]$) if $a_{ij} = 0$ for all $1 \leq i, j \leq m$ with $i \neq j$



$O_{m \times m}$ is a diagonal matrix

E_m is a diagonal matrix

Lemma 1.2 (補題)

For $A = [a_{ij}]$ and diagonal matrix $B = [b_{ij}]$

$AB = [a_{ij} b_{jj}]$ $BA = [a_{ij} b_{ii}]$

Proof Suppose $A = [a_{ij}]$ $B = [b_{ij}]$ $b_{ij} = 0$ for all $1 \leq i, j \leq m$ with $i \neq j$

Vectors

A $1 \times m$ matrix A is called a (row) vector (an m-dimensional)

$[a_1 \cdots\cdots a_m]$ a (row) vector

is denoted by bold small letters like $\mathbf{a}$ $\mathbf{b}$ $\mathbf{c}$

$\mathbf{v}$ $\mathbf{w}$ $\mathbf{u}$ etc.

An $m \times 1$ matrix A is called a (column) vector (an m-dimensional)

also denoted by $\mathbf{a}$ $\mathbf{b}$ $\mathbf{c}$ ⋯ $\mathbf{u}$ $\mathbf{v}$ $\mathbf{w}$ etc.

$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$

$m \times n$ matrix and n-dimensional column matrix can be multiplied to obtain an m-dimensional column matrix