Recap:

A, B $m\times n$ - matrices with $A = [a_{ij}]$, $B = [b_{ij}]$:

$$A+B := [a_{ij} + b_{ij}]$$

A is $l \times m$ - matrix with $A = [a_{ij}]$

B is $m \times n$ - matrix with $B = [b_{jk}]$

Then

$$AB = \left[\sum_{j=1}^{m} a_{ij} b_{jk}\right]_{\substack{1 \le i \le n \\ 1 \le j \le l}} \quad \longleftarrow l \times n \text{ matrix}$$

Ex)

$$\overbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}}^{3}_{\substack{\| \\ [a_{ij}]}} 3\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}_{\substack{\| \\ [b_{jk}]}} \right.$$

$$= \begin{bmatrix} \sum_{j=1}^{3} a_{1j} b_{j1} = 1\times 0 + 1\times 1 + 0\times 2 = 1 & \sum_{j=1}^{3} a_{1j} b_{j2} = 3 \\ \sum_{j=1}^{3} a_{2j} b_{j1} = 0\times 0 + 1\times 1 + (-1)\times 2 = -1 & \circ \\ \sum_{j=1}^{3} a_{3j} b_{j1} = 1\times 0 + 2\times 1 + 1\times 2 = 4 & \circ \end{bmatrix}$$

0 - matrices O

unit matrices E

$A + B = A + B$

$A + O = O + A = A$

$AE = EA = A$

$OA = AO = O$

scalar multiplication

Skalar スカラー

(German) (Japanese)

For a (real) number a

an $m\times n$ matrix $A = [a_{ij}]$

$$\underset{\text{scaler}}{a} A = [a a_{ij}]$$

Example

$$3 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \\ 21 & 24 & 27 \end{bmatrix}$$

eigen-value

Eigenwert

Theorem 2.1 (Associativity of addition and multiplication of matrices)

(1) $(A+B)+C = A+(B+C)$

(2) $(AB)C = A(BC)$

proof (1): Suppose $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$

$$\begin{aligned} (A+B)+C &= [a_{ij}+b_{ij}] + [c_{ij}] \\ &= [(a_{ij}+b_{ij}) + c_{ij}] \\ &= [a_{ij} + (b_{ij}+c_{ij})] \\ &= [a_{ij}] + [b_{ij}+c_{ij}] \\ &= A + (B+C) \end{aligned}$$

(2) : Suppose $A=[a_{ij}]_{\substack{1\le i\le m\\1\le j\le n}}$ $B=[b_{jk}]_{\substack{1\le j\le n\\1\le k\le p}}$ $C=[c_{kl}]_{\substack{1\le k\le p\\1\le l\le q}}$

The (i,l) entry of $(AB)C$ ← map matrix

$$= \sum_{k=1}^{p} \left(\text{the } (i,k) \text{ entry of } AB\right) c_{kl}$$

$$= \sum_{k=1}^{p} \underbrace{\left(\sum_{j=1}^{n} a_{ij} b_{jk}\right) c_{kl}}$$

$$(a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots) c_{kl}$$
$$= a_{i1}b_{1k}c_{kl} + a_{i2}b_{2k}c_{kl} + \cdots$$
$$= \sum_{j=1}^{n} a_{ij} b_{jk} c_{kl}$$

$$= \sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij} b_{jk} c_{kl}\right)$$

$$(a_{i1}b_{11}c_{1l} + a_{i2}b_{21}c_{1l} \cdots) \leftarrow k=1$$
$$+ (a_{i1}b_{12}c_{2l} + \cdots) \leftarrow k=2$$
$$+ (\qquad)$$

$$= \sum_{\substack{1\le k\le p\\1\le m\le n}} a_{ij} b_{jk} c_{kl}$$

$$= \sum_{j=1}^{m} \left(\sum_{k=1}^{p} a_{ij} b_{jk} c_{kl}\right)$$

$$= \sum_{j=1}^{n} a_{ij} \left(\underbrace{\sum_{k=1}^{p} b_{jk} c_{kl}}_{(i,l)\text{-entry of } BC}\right)$$

$= (i,l)$ entry of $A(BC)$

Thus $(AB)C = A(BC)$. ■

$(AB)C = (AC)B = A(CB)$

not always true !!! true

$\parallel$

$A(BC)$

Theorem 2.7 (Distributivity of multiplication, addition an scalar multiplication of matrices)

(1) $a(A+B) = aA + bA$

(2) $(a+c)A = aA + cA$

(3) $aAB = (aA)B = A(aB)$

(4) $A(B+C) = AB + AC$

(5) $(A+B)C = AC + BC$

Proof I will check (1) (3) (4) The rest is an exercise for you.

(4).

$A=[a_{ij}]$ $B=[b_{jk}]$ $C=[c_{jk}]$

The (i,k)-entry of $A(B+C)$

$= \sum_j a_{ij}$ (the j-k entry of $B+C$)

$= \sum_j a_{ij}(b_{jk}+c_{jk})$

$= \sum_j (a_{ij}b_{jk} + a_{ij}c_{jk}) = \left(\sum_j a_{ij}b_{jk}\right) + \left(\sum_j a_{ij}c_{jk}\right)$

= (i,k)-entry of AB + (i,k)-entry of AC

= (i,k)-entry of $(AB+AC)$

Thus $A(B+C) = AB+AC$. $\boxtimes$

$\mathbb{N}$ $\mathbb{R}$

$\mathbb{R}$ $\mathbb{N} \xrightarrow{1\text{-}1} \mathbb{Q} = \{\frac{n}{m} \mid n, m \in \mathbb{N}, m \neq 0\}$

there is no 1-1 $\mathbb{N} \to \mathbb{R}$

Transposition of matrices

For an $m \times n$ matrix $A = [a_{ij}]_{\substack{1\le i\le m \\ 1\le j\le n}}$

The transposition of A (notation: ${}^tA$) is the $n \times m$ matrix $[b_{ji}]$ where

$b_{ji} = a_{ij}$

Example

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$ ${}^tA = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$

转置

Theorem 23

${}^t({}^tA) = A$

$[1\ 2\ 3]\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 22$

Suppose a and b are n dimensional column vectors

$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ ab is not defined

but we can calculate

$({}^ta)b$

$[a_1\ a_2 \cdots a_n]\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \left[\sum_{i=1}^n a_i b_i\right]$

$= \sum a_i b_i$

identified!