Determinant of square matrices

For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = [a_1\, a_2]$, $\quad a_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$, $a_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$

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the determinant (notation: $|A|$

$\det(A)$ $\quad \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix}$

or $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{vmatrix}$)

is the scalar $a_{11}a_{22} - a_{12}a_{21} \in \mathbb{R}$

<u>Thm 5.1</u>

(1) 2×2 matrix $A = [a_1\, a_2]$ and $c \in \mathbb{R}$

$c\det(A) = \det([ca_1\ a_2]) = \det([a_1\ ca])$,

(2) If $a_1 = a_1^0 + a_1^1$ then

$\det([a_1\ a_2]) = \det([(a_1^0 + a_1^1)\ a_2])$

$= \det([a_1^0\ a_2]) + \det([a_1^1\ a_2])$

Similarly if $a_2 = a_2^0 + a_2^1$ then

$\det([a_1\ a_2]) = \det([a_1\ (a_2^0 + a_2^1)])$

$= \det([a_1\ a_2^0]) + \det([a_1\ a_2^1])$

(3) For 2×2 matrices

$\det(AB) = \det(A)\det(B)$

(4) $\det({}^tA) = \det(A)$

<u>proof</u>

(1): $\det([ca_1\ a_2])$

$= \det\left(\begin{bmatrix} ca_{11} & a_{12} \\ ca_{21} & a_{22} \end{bmatrix}\right)$

$= ca_{11}a_{22} - ca_{12}a_{21}$

$= c(a_{11}a_{22} - a_{12}a_{21}) = *$

$* = c\det([a_1\ a_2])$

Similarly

$\det([a_1\ ca_2])$

$= \det\left(\begin{bmatrix} a_{11} & ca_{12} \\ a_{21} & ca_{22} \end{bmatrix}\right)$

$= \cdots = c\det([a_1 a_2])$

(4):

$\det({}^tA) = \det\left(\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\right)$

$= a_{11}a_{22} - a_{12}a_{21}$

$= \det(A)$.

The rest can be done similarly (with a slightly heavier calculation!) $\boxtimes$

R_θ rotation (anticlockwise by angle θ)

$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

<u>Lemma 5.2</u> For $n\times n$ matrices A and B

if $B = [b_1 \cdots b_n]$

then

$AB = [Ab_1 \cdots Ab_n]$

n-dimensional column vectors

<u>proof</u> Let $A = [a_{ij}]$ $B = [b_{jk}]$. Then

$b_k = \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix}$ $\quad$ (i,j)-entry of $AB = \sum_{j=1}^{n} a_{ij}b_{jk}$

: The component of Ab_k

$\boxtimes$

Thm 5.3 For a 2×2 matrix $A=[a_1\, a_2]$

(1) $R_\theta A = [R_\theta a_1\ R_\theta a_2]$

(2) $\det(R_\theta) = 1$

(3) $\det(R_\theta A) = \det(A)$

Proof (1) By Lemma 5.2.

(2) $\det(R_\theta) = \det\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right) = \cos\theta\cdot\cos\theta - (-\sin\theta)\sin\theta$

$= \cos^2\theta + \sin^2\theta$

$= 1$

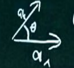
(3)

$$\det(R_\theta A) \overset{\text{Thm 5.2 (3)}}{=} \underbrace{\det(R_\theta)}_{1 \text{ by (2)}}\det(A)$$

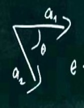
$$= \det(A)$$

Thm 5.4 For a 2×2 matrix $A=[a_1\, a_2]$

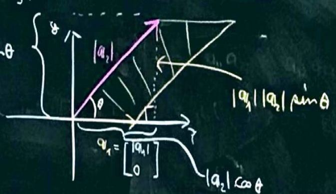
let θ be the angle of the vectors a_1 and a_2 where $|\theta|$ is the angle and

$\theta>0$ if a_2 is in anti clockwise position to a_1 : 

$\theta<0$ if a_2 is in clockwise position to a_1



$\det(A)$ is the area of the parallelogram spanned by a_1 and a_2 when we consider the area is negative if the angle θ is negative



Proof By Theorem 5.3 we may assume that the direction of a_1 coincides with the direction of x-axis of the plane

$$\det(A) = \det\left(\left[\begin{bmatrix}|a_1|\\0\end{bmatrix}\begin{bmatrix}|a_2|\cos\theta\\|a_2|\sin\theta\end{bmatrix}\right]\right) = |a_1||a_2|\sin\theta \quad \boxtimes$$

Cor 5.5 For 2×2 matrix $A=[a_1\, a_2]$

(1) If a_1 and a_2 are orthogonal to each other then $\det(A) = \pm|a_1||a_2|$ ($\pm$ depending on the position of a_2 to a_1)

(2) If a_2 is a multiple of a_1 then $\det(A) = 0$

Determinant of 3X3 matrices

For $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$= a_{11}\cdot\det\left(\underbrace{\begin{bmatrix} \square & \square \\ \square & a_{22} \end{bmatrix}}_{= a_{22}}\right) - a_{12}\det\left(\underbrace{\begin{bmatrix} \square & \square \\ a_{21} & \square \end{bmatrix}}_{= a_{21}}\right)$$

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\right) = a_{11}\cdot\det\left(\begin{bmatrix} \square & \square & \square & \square \\ \square & a_{22} & a_{23} & a_{24} \\ \square & a_{32} & a_{33} & a_{34} \\ \square & a_{42} & a_{43} & a_{44} \end{bmatrix}\right) - a_{12}\det\left(\begin{bmatrix} \square & \square & \square & \square \\ & \square & & \\ & \square & & \\ & \square & & \end{bmatrix}\right)$$

$$\det(A) = a_{11}\det\left(\underbrace{\begin{bmatrix} \square & \square & \square \\ \square & a_{22} & a_{33} \\ \square & a_{32} & a_{33} \end{bmatrix}}_{= a_{11}\cdot\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}}\right) - a_{12}\det\left(\underbrace{\begin{bmatrix} \square & \square & \square \\ a_{21} & \square & a_{23} \\ a_{31} & \square & a_{33} \end{bmatrix}}_{= a_{12}\cdot\det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right)}\right) + a_{13}\det\left(\underbrace{\begin{bmatrix} \square & \square & \square \\ a_{21} & a_{22} & \square \\ a_{31} & a_{32} & \square \end{bmatrix}}_{= a_{13}\cdot\det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right)}\right)$$

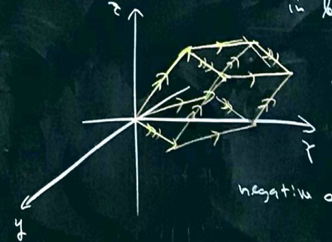
$$+ a_{13}\det\left(\begin{bmatrix} \square & \square & \square & \square \\ & & \square & \\ & & \square & \\ & & \square & \end{bmatrix}\right) - a_{14}\det\left(\begin{bmatrix} \square & \square & \square & \square \\ & & & \square \\ & & & \square \\ & & & \square \end{bmatrix}\right)$$

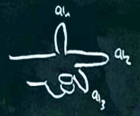
Thm 5.6

For a 3x3 matrix $A=[a_1\, a_2\, a_3]$

$|\det A|$ is the volume of the parallelepiped spanned by the vectors $a_1\ a_2\ a_3$

$\det(A)$ is positive if $a_1\ a_2\ a_3$ build in left hand system





negative otherwise.

Sarrus calculation rule

For $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

Thm 5.7 $A=[a_1\, a_2\, a_3]$ 3x3 matrix

(1) $c\det(A) = \det([ca_1\, a_2\, a_3])$
$= \det([a_1\, ca_2\, a_3]) = \det([a_1\, a_2\, ca_3])$

(2) If $a_1 = a_1^0 + a_1^1$ then
$\det(A) = \det([a_1^0\, a_2\, a_3]) + \det([a_1^1\, a_2\, a_3])$

....

$$\det(A) = \underbrace{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}} - \big(\underbrace{a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}}\big)$$