Solution of system of linear equations

Example

$3x^2 + x + y^4 + y^3 = 7$

$\sin(x+\pi) = \frac{3}{4}\pi$

$\vdots$

not linear

$$(*)\begin{cases} 3x + y - 7z = 0 \\ 4x - y - z = 5 \\ x - y + 2z = 2 \end{cases}$$

Problem: Find (a) triple(s) of numbers $a, b, c \in \mathbb{R}$ s.t. they satisfy all the equations if they are substituted in variables x, y, z resp.

$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is said to be a solution of the system $(*)$.

3 Scenarios: (1) there is a unique solution to a system of lin. equations.

(2) there are infinitely many solutions to a system of lin. equations

(3) There is no solution to a system of lin. eq.

Example

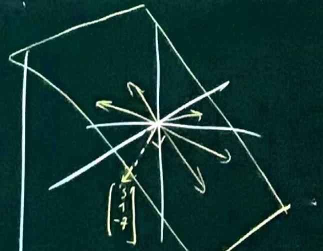
$$\begin{cases} 3x_1 + 4x_2 - 2x_3 + \sqrt{5}\,x_4 = 7 \\ x_1 \underset{\substack{\| \\ +0x_2}}{\bigcirc} + 3x_3 \underbrace{- 4x_4}_{+(-4)x_4} = -2 \end{cases}$$

Explanation in case of 3 unknown variables

Linear equations in x, y, z represent a plane in $\mathbb{R}^3$

For example, $3x + y - 7z = 0 \iff \begin{bmatrix} 3 \\ 1 \\ -7 \end{bmatrix} \overset{\text{inner product}}{\cdot} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\iff$ the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is diagonal to $\begin{bmatrix} 3 \\ 1 \\ -7 \end{bmatrix}$



$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R} \;\middle|\; 3x + y - 7z = 0 \right\}$

= the vectors on the plane to which the vector $\begin{bmatrix} 3 \\ 1 \\ -7 \end{bmatrix}$ is perpendicular.

The solutions of a system of linear equations with $\underbrace{\text{variables}}_{3 \text{ unknown}}$ are the points in the intersection of three planes represented by the linear equations.

Solution of system of linear equations

Example

$$(*)\begin{cases} 3x + y - 7z = 0 \\ 4x - y - z = 5 \\ x - y + 2z = 2 \end{cases}$$

$$\left.\begin{array}{l} 3x^2 + x + y^4 + y^3 = 7 \\ \sin(x+\pi) = \frac{3}{4}\pi \\ \vdots \end{array}\right\} \text{not linear}$$

Problem Find (a) triple(s) of numbers $a, b, c \in \mathbb{R}$ s.t. they satisfy all the equations

Representation of systems of linear equations by matrices or vectors

Example (*) can be represented by

$$\begin{bmatrix} 3 & 1 & -7 \\ 4 & -1 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$$

↑ coefficient matrix of the system (*)

If A is $n \times n$ matrix, $b \in \mathbb{R}^n$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

and if A has the inverse matrix A^{-1}

(i.e. a matrix B s.t. $BA = AB = E_n$)

Then the condition $Ax = b$ is equivalent

to $\underbrace{A^{-1}Ax}_{= E_n x = x} = A^{-1}b$ So in this case $Ax = b$ has the unique solution $x = A^{-1}b$

The coefficient matrix augmented (or enlarged) by the vector of the constants on the right side of equations is called the augmented coefficient matrix.

Example:

$$\left[\begin{array}{ccc:c} 3 & 1 & -7 & 0 \\ 4 & -1 & -1 & 5 \\ 1 & -1 & 2 & 2 \end{array}\right]$$

Two systems of linear equations $Ax = b$ and $A'x = b'$ are said to be equivalent if they have the same solution set. i.e.

$$\{a \in \mathbb{R}^3 \mid Aa = b\} = \{a \in \mathbb{R}^3 \mid A'a = b'\}$$

Gaussian elimination method

Given a linear equation $Ax = b$ we transform the augmented coefficient matrix $[A \,\vdots\, b]$ to a matrix of the form

echelon form

which corresponds to a system of linear equations equivalent to the original system of linear equations

Gaussian elimination

Example

$$(*)\begin{cases} 3x + y - 7z = 0 \\ 4x - y - z = 5 \\ x - y + 2z = 2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & -7 & 0 \\ 4 & -1 & -1 & 5 \\ 1 & -1 & 2 & 2 \end{array}\right]$$ switch ① row and ③ row (① ↷ ③)

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 4 & -1 & -1 & 5 \\ 3 & 1 & -7 & 0 \end{array}\right]$$ ② + (−4)×① ; ③ + (−3)×①

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & -9 & -3 \\ 0 & 4 & -13 & -6 \end{array}\right]$$ ② × $\frac{1}{3}$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 4 & -13 & -6 \end{array}\right]$$ ③ + (−4)×②

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

$$\begin{cases} x - y + 2z = 2 \\ y - 3z = -1 \\ -z = -2 \end{cases}$$

$z = 2$

$y - 3\cdot 2 = -1 \Rightarrow y = -1 + 6 = 5$

$x - 5 + 2\cdot 2 = 2 \Rightarrow x = 2 + 5 - 4 = 3$

■ : non zero entry □ : any entry which can be also non zero

$$\begin{bmatrix} ■ & □ & □ & □ \\ 0 & ■ & □ & □ \\ 0 & 0 & ■ & □ \end{bmatrix}$$ unique solution

$$\begin{bmatrix} ■ & □ & □ & □ \\ 0 & 0 & ■ & □ \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ infinitely many solutions

$$\begin{bmatrix} ■ & □ & □ & □ \\ 0 & 0 & ■ & □ \\ 0 & 0 & 0 & ■ \end{bmatrix}$$ no solution !

$0x + 0y + 0z = ■$

Basic transformations
- switching two rows
- add multiple of one row to another row
- replace a row with non zero multiple of the row