Recap: For an n×n-matrix A an n×n-matrix B is said to be a(the) inverse of A iff

$$BA = BA = E_n = \left.\underbrace{\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}}_{n}\right\}n$$

- B is unique (Lemma 1.2)

A is said to be invertible if A has the inverse

The inverse of A is denoted by A^{-1}. We have

Lemma 2.1 If A is invertible then A^{-1} is also invertible and $(A^{-1})^{-1} = A$

- proof We have $A(A^{-1}) = (A^{-1})A = E_n$

Thus A is an inverse of A^{-1}

But by Lemma 1.2 such inverse is unique. ☑

Lemma 2.2 For n×n-matrices A, B if A and B are both invertible then AB is invertible

- proof We show that $B^{-1}A^{-1}$ is the inverse of AB.

$$(B^{-1}\underbrace{A^{-1})(A}_{}B) = B^{-1}\underbrace{(A^{-1}A)}_{E_n}B = B^{-1}(E_nB)$$

$$= B^{-1}B = E_n$$

$$(AB)(B^{-1}A^{-1}) = \cdots = E_n$$

☒

Cor 2.3 If $A_1, \dots A_n$ are invertible n×n-matrix then $A_1A_2 \cdots A_m$ is invertible and $A_m^{-1}A_{m-1}^{-1} \cdots A_1^{-1}$ is the inverse of $A_1A_2 \cdots A_m$.

Lemma 2.4 For n×n matrices A, B. B is the inverse of A if we have one of the condition $BA = E_n$ and $AB = E_n$

The proof is postponed.

$$\begin{cases}2x+3y-z=-3\\ -x+2y+2z=1\\ x+y-z=-2\end{cases}$$

$$\begin{cases}x+y-z=-2\\ y+z=1\\ \boxed{-2z=-4}\end{cases}$$

$z=2$

$y=-1$

$x=1$

Recap: Example of $\underbrace{\text{Gaussean elimination}}_{\text{the application of the}}$ method:

$$A=\left[\begin{array}{ccc:c}2 & 3 & -1 & -3\\ -1 & 2 & 2 & 1\\ 1 & 1 & -1 & -2\end{array}\right] ①\leftrightarrow③$$

$$\left[\begin{array}{ccc:c}1 & 1 & -1 & -2\\ -1 & 2 & 2 & 1\\ 2 & 3 & -1 & -3\end{array}\right] \begin{matrix}②+①\\ ③-2①\end{matrix}$$

$$\left[\begin{array}{ccc:c}1 & 1 & -1 & -2\\ 0 & 3 & 1 & -1\\ 0 & 1 & 1 & 1\end{array}\right] ②\leftrightarrow③$$

$$\left[\begin{array}{ccc:c}1 & 1 & -1 & -2\\ 0 & 1 & 1 & 1\\ 0 & 3 & 1 & -1\end{array}\right] ③-3\times②$$

$$\left[\begin{array}{ccc:c}1 & 1 & -1 & -2\\ 0 & 1 & 1 & 1\\ 0 & 0 & -2 & -4\end{array}\right]$$

$$\underbrace{T_{13}^{3}}_{\text{invertible}}A$$

matrices for elementary row operations.

$$\underbrace{L_{13}^{3}(-2)L_{12}^{3}(1)T_{13}^{3}}_{\substack{\text{invertible}\\ \text{by Cor 2.3}}}A$$

$\vdots$

Lemma 2.5 For $n \times n$ matrix A, and the system of lin. equations $Ax = b$ where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, If B is an invertible $n \times n$ matrix, then

$$Ax = b \text{ is equivalent to } (BA)x = Bb$$

proof We have to show, for any n-dim vector $a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

a is a solution of $Ax = b$ $\Leftrightarrow$ a is a solution of $(BA)x = Bb$.

Suppose first that a is a solution of $Ax = b$. This means that we have $Aa = b$. Multiplying both sides of this equation by B we get $(BA)a = Bb$. This shows that a is a solution of $(BA)x = Bb$.

Suppose now that a is a solution of $(BA)x = Bb$. This means that we have $(BA)a = Bb$. Multiplying both sides of this equation by $\underbrace{B^{-1}}_{\text{from the left}}$ we obtain

$$\underline{\underline{Aa}} = (\underbrace{B^{-1}B}_{E_n})Aa = B^{-1}(BA)a = B^{-1}Bb = (\underbrace{B^{-1}B}_{E_n})b = \underline{\underline{b}}$$

Thus a is a solution of $Ax = b$. ☒

Continuation of the previous example

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -4 \end{array}\right] ① - ②$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -4 \end{array}\right] -\tfrac{1}{2} \times ③$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} ① + 2 \times ③ \\ ② - ③ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x = 1$
$y = -1$
$z = 2$

$$Ax = b \rightsquigarrow E_n x = (\cdots L^3_{13}(-2) L^3_{12}(1) T^3_{13})b$$

$$\underline{(\cdots L^3_{12}(-1) L^3_{12}(1) T_{13})A} = E_n$$

By Lemma 2.4 it follows that

$$\underline{\underline{(\cdots L^3_{12}(-2) L^3_{12}(1) T^3_{13}) = A^{-1}}} \quad !!$$

$$\begin{cases} 2x + 3y - z = -3 \\ -x + 2y + 2z = 1 \\ x + y - z = -2 \end{cases}$$

$$\begin{cases} x + y - z = -2 \\ y + z = 1 \\ -2z = -4 \end{cases}$$

$z = 2$
$y = -1$
$x = 1$

$Ax = b$

Recap: Example of $\underbrace{\text{Gaussian elimination}}_{\text{the application of the}}$ method:

$$\tilde{A} = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{} & b \\ 2 & 3 & -1 & -3 \\ -1 & 2 & 2 & 1 \\ 1 & 1 & -1 & -2 \\ \multicolumn{3}{c|}{A} & \end{array}\right] ① \leftrightarrow ③$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ -1 & 2 & 2 & 1 \\ 2 & 3 & -1 & -3 \end{array}\right] \begin{array}{l} ② + ① \\ ③ - 2 \times ① \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 3 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right] ② \leftrightarrow ③$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 1 & -1 \end{array}\right] ③ - 3 \times ②$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

$$\underbrace{T^3_{13}}_{\text{invertible}} \tilde{A}$$

matrices for elementary row operations.

$$\underbrace{L^3_{13}(-2) L^3_{12}(1) T^3_{13}}_{\text{invertible by Cor 2.3}} \tilde{A}$$

[illegible] x

$$T^3_{13} A x = T^3_{13} b$$

A method for the calculation of the inverse of matrix

To calculate the inverse of the matrix $A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 2 & 2 \\ 1 & 1 & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \quad ① \leftrightarrow ③$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & 1 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} ② + ① \\ ③ - 2 \times ① \end{array} \qquad T^3_{13}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & -2 \end{array}\right] \quad ② \leftrightarrow ③ \qquad L^3_{13}(-2) L^3_{12}(1) T^3_{13}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & -2 \\ 0 & 3 & 1 & 0 & 1 & 1 \end{array}\right] \qquad T^3_{23} L^3_{13}(-2) L^3_{12}(1) T^3_{13} !$$

$$\vdots$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \end{array}\right]$$

$$Ax = b \rightsquigarrow E_n x = (\cdots L^3_{13}(-2) L^3_{12}(1) T^3_{13}) b$$

$$E_n = \underline{(\cdots L^3_{13}(-2) L^3_{12}(1) T^3_{13}) A}$$

By Lemma 2.4 it follows that

$$\underline{\underline{(\cdots L^3_{13}(-2) L^3_{12}(1) T^3_{13}) = A^{-1}}} \quad !!$$

$$\begin{cases} 2x + 3y - z = -3 \\ -x + 2y + 2z = 1 \\ x + y - z = -2 \end{cases}$$

$$\begin{cases} x + y - z = -2 \\ y + z = 1 \\ \boxed{-2z = -4} \end{cases}$$

$z = 2$
$y = -1$
$x = 1$

Recap: Example of $\underbrace{\text{Gaussean elimination}}_{\text{the application of the}}$ method:

$Ax = b$

$$\tilde{A} = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{} & \overset{b}{} \\ 2 & 3 & -1 & -3 \\ -1 & 2 & 2 & 1 \\ 1 & 1 & -1 & -2 \end{array}\right] \quad ① \leftrightarrow ③$$

$\underbrace{}_{A}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ -1 & 2 & 2 & 1 \\ 2 & 3 & -1 & -3 \end{array}\right] \quad \begin{array}{l} ② + ① \\ ③ - 2① \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 3 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right] \quad ② \leftrightarrow ③$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 1 & -1 \end{array}\right] \quad ③ - 3 \times ②$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

$$\underbrace{T^3_{13}}_{\text{invertible}} \tilde{A}$$

matrices for elementary row operations.

$$\underbrace{L^3_{13}(-2) L^3_{12}(1) T^3_{13}}_{\text{invertible by Co 2.3}} \tilde{A}$$

$$\vdots$$

corresponds to

$$T^3_{13} A x = T^3_{13} b$$