Inverse of square matrices

Example of calculation of inverse

$$A=\begin{bmatrix}1&2&1\\2&5&2\\1&3&2\end{bmatrix}$$

Pivot

$$\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\2&5&2&0&1&0\\1&3&2&0&0&1\end{array}\right]\begin{array}{l}\\ ②-2\times①\\ ③-①\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&0&-2&1&0\\0&1&1&-1&0&1\end{array}\right]\begin{array}{l}①-2\times②\\ \\ ③-②\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&1&5&-2&0\\0&1&0&-2&1&0\\0&0&1&1&-1&1\end{array}\right]\begin{array}{l}①-③\\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&4&-1&-1\\0&1&0&-2&1&0\\0&0&1&1&-1&1\end{array}\right]$$

Thus $\begin{bmatrix}4&-1&-1\\-2&1&0\\1&-1&1\end{bmatrix}$ is the inverse of A

Check:

$$\begin{bmatrix}1&2&1\\2&5&2\\1&3&2\end{bmatrix}\begin{bmatrix}4&-1&-1\\-2&1&0\\1&-1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} //$$

$$B=\begin{bmatrix}1&0&2\\3&2&4\\0&1&-1\end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\3&2&4&0&1&0\\0&1&-1&0&0&1\end{array}\right]\begin{array}{l}\\ ②-3\times①\\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&2&-2&-3&1&0\\0&1&-1&0&0&1\end{array}\right]\ ②\leftrightarrow③$$

$$\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&-1&0&0&1\\0&2&-2&-3&1&0\end{array}\right]\begin{array}{l}\\ \\ ③-2\times②\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&\boxed{2}&1&0&1\\0&1&\boxed{-1}&0&0&1\\0&0&0&-3&1&-2\end{array}\right]$$

There is no inverse to B

i.e. B is not invertible!

$$\begin{cases}x+2y+z=2\\2x+5y+2z=-1\\x+3y+2z=1\end{cases} \iff A\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=A^{-1}\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

has the unique solution

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}8\\-5\\4\end{bmatrix} //$$

$$\begin{bmatrix}4&-1&-1\\-2&1&0\\1&-1&1\end{bmatrix}\begin{bmatrix}2\\-1\\1\end{bmatrix}=\begin{bmatrix}8\\-5\\4\end{bmatrix}$$

Lemma 3.1 For an $n \times n$-matrix A

A is invertible $\Leftrightarrow$ ${}^tA$ is invertible

and $({}^tA)^{-1} = {}^t(A^{-1})$

${}^tA\,{}^tB = {}^t(BA)$

Proof Suppose that A is invertible. Then

$${}^tA\,{}^t(A^{-1}) = {}^t(\underbrace{A^{-1}A}_{E_n}) = {}^tE_n = E_n$$

Similarly ${}^t(A^{-1})\,{}^tA = E_n$

Thus ${}^t(A^{-1})$ is the inverse of ${}^tA$ and ${}^tA$ is invertible. The other direction also holds since

${}^t({}^tA) = A$. ☒

Lemma 3.2 (proved later)

For an $n \times n$ matrix $A = [a_1 \cdots a_n]$ t.f.a.e. ← the following are equivalent

(a) A is invertible

(b) $a_1 \cdots a_n$ are linearly independent

(i.e. for any $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$ if $\alpha_1 a_1 + \cdots + \alpha_n a_n = 0$ then $\alpha_1 = \cdots = \alpha_n = 0$)

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix}$$

Cor 3.3 For an $n \times n$ matrix $A = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ t.f.a.e.

b_i : n-dimensional row vector

(a) A is invertible

(b) $b_1 \cdots b_n$ are linearly independent

(i.e., for any $\beta_1, \cdots, \beta_n \in \mathbb{R}$ if $\beta_1 b_1 + \cdots + \beta_n b_n = 0$ then $\beta_1 = \cdots = \beta_n = 0$)

n-dimensional zero row vector: $0 = \underbrace{[0 \cdots 0]}_{n}$

Proof By Lemma 3.2 and Lemma 3.1 ☒

B is not invertible since

$[3\ 2\ 4] = 3[1\ 0\ 2] + 2[0\ 1\ -1]$

or

$-3[1\ 0\ 2] + 1[3\ 2\ 4] - 2[0\ 1\ -1] = [0\ 0\ 0]$

Thus $[1\ 0\ 3]$, $[3\ 2\ 4]$, $[0\ 1\ -1]$ are not lin. independent. By Cor 3.3 B is not invertible.

General theory of determinants

Objective : Prove, for $n\times n$ matrix A

A is invertible $\Leftrightarrow$ $\det(A) \neq 0$

We (re) introduce determinants using the notion of permutations defined below

For $n \in \mathbb{N}$, $n \geq 1$ a permutation π (of degree n ?) is a 1-1 onto mapping $\pi : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$

$S_n = \{\pi \mid \pi \text{ is a permutation of degree } n\}$

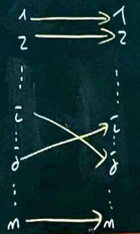
↑ S for "symmetric group"

- $\mathrm{id}_{\{1,\dots,n\}} : \{1,\dots,n\} \to \{1,\dots,n\}$; $i \mapsto i$ is a permutation. or simply id if it is clear which n is meant!
- For $1 \leq i \leq j \leq n$

$(i\ j) : \{1,\dots,n\} \to \{1,\dots,n\}$ is defined by

$$(i\ j)(k) = \begin{cases} j & \text{if } k=i \\ i & \text{if } k=j \\ k & \text{otherwise} \end{cases}$$

transposition   if $i=j$ then $(i\ i) = \mathrm{id}$



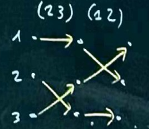
If $\pi, \tau \in S_n$ then

$\tau\pi : \{1,\dots,n\} \to \{1,\dots,n\}$; $i \mapsto \tau(\pi(i))$

$\tau\pi$ is the just the $\underbrace{\text{composition of functions } \tau \text{ and } \pi}_{\tau\circ\pi}$

This multiplication is commutative :

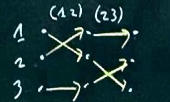
$\sigma_1 = (1\ 2)(2\ 3)$



$\sigma_1(1) = 2$
$\sigma_1(2) = 3$
$\sigma_1(3) = 1$

$\sigma_1 \neq \sigma_2$

$\sigma_2 = (2\ 3)(1,2)$



$\sigma_2(1) = 3$
$\sigma_2(2) = 1$
$\sigma_2(3) = 2$

An extra exercise session (補講)

July 10 (wed) 17^{00} –

in K403