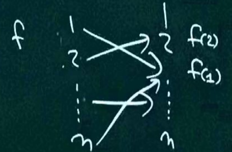
Recap: $f : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ is a permutation of degree n if f is 1-1 and onto



f is represented as $\begin{pmatrix} 1 & \cdots & n \\ f(1) & \cdots & f(n) \end{pmatrix}$

$S_n = \{ f : f \text{ is a permutation of degree } n \}$

↑ symmetric group

For $\sigma, \tau \in S_n$ $\tau\sigma$ the composition of σ and τ

$\tau\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$; $\tau\sigma(i) = \tau(\sigma(i))$ $\quad \tau\sigma \in S_n$

$id_n : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$; $i \longmapsto i$ $\quad id_n \in S_n$

For $i, j \in \{1, \dots, n\}$ $(i, j) : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$

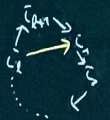
$$(i, j)(k) = \begin{cases} i & \text{if } k = j \\ j & \text{if } k = i \\ k & \text{otherwise} \end{cases}$$

↗ transposition

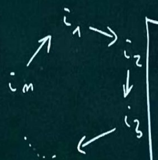
$\sigma = (i_1 \, i_2 \, \cdots \, i_n)$

$$\sigma(k) = \begin{cases} i_{l+1} & \text{if } k = i_l \text{ for some } 1 \le l < n \\ i_1 & \text{if } k = i_n \\ k & \text{otherwise} \end{cases}$$

where $i_1 \dots i_n$ are pairwise distinct



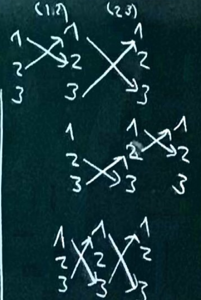
cyclic permutation



Lemma 4.1 For any n every cyclic permutation (in S_n) can be represented as a product of some number of transpositions.

Example

$(1\,2\,3) = (1\,3)(1\,2)$
$= (1\,2)(2\,3)$
$= (1\,3)(1\,3)(1\,2)(2\,3)$



$id = (1\,3)(1\,3)$

Proof By induction on k

we show: (*) cyclic permutation of the form $(i_1 \cdots i_k)$ can be represented as a product of transpositions

For $k = 2$ ok since cyclic permutation of cycle 2 is a transposition.

Suppose that (*) holds for $k = l$. For $l + 1$:

Let $(i_1, \dots, i_l\, i_{l+1})$ be a cyclic permutation of cycle $l+1$. Then

$(i_1 \cdots i_l)(i_l\, i_{l+1}) = (i_1 \dots i_{l+1})$ Thus $(i_1 \cdots i_{l+1})$ can be represented as a product of transpositions.

$\sigma_1 \cdots \sigma_m$ ← transpositions

Lemma 4.2 Any permutation can be represented as a product of cyclic permutations. in S_n, $n \geq 2$

Proof By induction on n. If $n = 2$

Then $S_2 = \{ id_2, (1\,2) \}$ $\qquad$ $1 \to 1$, $2 \to 2$ $\qquad$ $1 \to 2$, $2 \to 1$

$id_2 = (1\,2)(1\,2)$

$(1\,2) = (1\,2)$

Thus the statement of Lemma holds for S_n

Suppose now that the statement of the Lemma holds for all $n < m$ and show that the statement also holds for $n = m$. $m > 2$

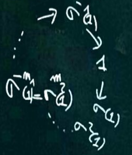
Let $\sigma \in S_m$ $\qquad$ $\sigma^{-1}(\sigma^{-1}(1))$ $\shortparallel$

Consider the sequence $\quad \cdots \sigma^{-2}(1)\ \sigma^{-1}(1)\ 1\ \sigma(1)\ \sigma^2(1) \cdots\cdots$

Let m_1 and m_2 be minimal s.t. $\sigma^{-m_1}(1) = \sigma^{m_2}(1)$

Let $X = \{ \sigma^{-m_1}(1), \sigma^{-m_1-1}(1), \ldots, \sigma^{-1}(1), 1, \sigma(1), \sigma^2(1), \ldots \}$

And $Y = \{1, \ldots, m\} \setminus X$



$\sigma \restriction Y$ $\qquad$ Since $|Y| < m$

The mapping σ restricted to Y

1-1 onto mapping on Y

$\sigma \restriction Y$ can be represented as a product of cyclic permutation, say

$\sigma \restriction Y = c_1 \cdots c_p$ where

$c_i : Y \to Y$ we can regard c_i as mapping from $\{1, \ldots, m\}$ to $\{1 \ldots m\}$

Then. $\sigma = \sigma \restriction X\ c_1 \cdots c_p$

$= c_1 \cdots c_p\, \sigma \restriction X$ $\quad$ ∎

Lemma 4.3 All permutations can be represented as a product of transpositions.

Proof By Lemma 4.1 and Lemma 4.2. ∎

Lemma 4.4 Suppose that $\tau \in S_n$ is represented as

$\tau = \tau_1 \cdots \tau_m$ $\quad$ where $\tau_1 \ldots \tau_m$, $\sigma_1 \ldots \sigma_n$ are transpositions

$= \sigma_1 \cdots \sigma_n$

Then $(-1)^m = (-1)^n$ i.e. the parity (oddness or evenness) of m and n is the same.

Proof See the text book 問題 3.1 5.−8.

For $\tau \in S_n$ with $\tau = \tau_1 \cdots \tau_m$ for transpositions $\tau_1 \ldots \tau_m$

The number $(-1)^m$ does not depend on the representation by Lemma 4.4.

This number is denoted as $sgn(\sigma)$ and called signature of σ.

For $n \times n$ matrix $A = [a_{ij}]$ we redefine the determinant of A as

$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma)\, a_{1,\sigma(1)}\, a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

Lemma 4.5 This definition coincides with the old definition of determinant.

Proof For 2×2 matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$ Noting,

$S_2=\{id, (1\,2)\}$, our present definition of the determinant gives: $sgn(id)=1$, $sgn((1\,2))=-1$

$$\det(A) = a_{1,id(1)}\, a_{2\,id(2)} + (-1)\, a_{1\,(1,2)(1)}\, a_{2\,(1,2)(2)}$$

$$= \boxed{a_{11}a_{22} - a_{12}a_{21}} = \text{the determinant of } A \text{ in the old definition.}$$

By induction the following Lemma shows that the determinant in the present definition is equal to the determinant in the old definition

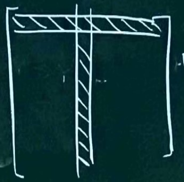
Lemma 4.6 (with the new definition of determinant)

We have: For $n\times n$ matrix A with $n>2$

$$\det A = \sum_{j=1}^{n} (-1)^{j+1} \det(A_{1j})$$

where A_{1j} is the $(n-1)\times(n-1)$ matrix consisting of entries a_{pq} of A where $p\neq 1$ and $q\neq j$

j



For $A_{ij} = [a'_{pq}]$

$$a'_{pq} := \begin{cases} a_{p+1\,q} & \text{if } q<j \\ a_{p+1\,q+1} & \text{if } j\leq j \end{cases}$$

Proof

$$\det(A) = \sum_{\sigma\in S_n} sgn(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

$= \text{- - -}$ to be continued in the next lecture

For $n=3$ if $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ $S_3=\{id, (1,2), (1\,3), (2\,3), (1,2,3)\ (1\,3\,2)\}$

$(1,2,3) = (1\,2)(2\,3)$, $(1\,3\,2) = (1\,3)(2\,3)$

$sgn(id)=1$ $sgn(1\,2\,3)=1$

$sgn(1\,3\,3)=1$ $sgn(1\,2)=sgn(1\,3)=sgn(2\,3)=-1$

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$-(\qquad\qquad)$$

The number $(-1)^m$ does not depend on the representation by Lemma 4.4.

This number is denoted as $sng(\sigma)$ and called signature of σ.

For $n\times n$ matrix $A=[a_{ij}]$ we define the determinant of A as

$$\boxed{\det(A) = \sum_{\sigma\in S_n} sgn(\sigma)\, a_{1,\sigma(1)}\, a_{2,\sigma(2)} \cdot\ldots\cdot a_{n,\sigma(n)}}$$

Lemma 4.5 This definition coincides with the old definition of determinant.