$n = 1, 2, 3 \cdots$

$S_n = \{\sigma : \sigma \xrightarrow{\text{bijections}} \{1, \dots, n\} \to \{1, \dots, n\}\}$

1-1, onto

symmetric group

permutation of degree n

For $1 \le i \le j \le n$ (i, j) denotes the element of S_n

transposition

(i, j) $1 \cdots i \quad j \cdots n$; $1 \cdots\cdots \quad \cdots\cdots$

$id \in S_n$ $\quad id : \{1, \dots, n\} \to \{1, \dots, n\}; \ i \mapsto i$

Lemma 4.4 + (4.5)

All $\sigma \in S_n$ can be represented as a product of transpositions.

If $\sigma = \tau_1 \cdots \tau_n = \tau'_1 \cdots \tau'_{n'}$

(it is possible that $n \ne n'$)

then the parity of them is the same.

(oddness or evenness)

In particular, $\underbrace{(-1)^n = (-1)^{n'}}_{\text{depends only on } \sigma}$ ← This is denoted by $\mathrm{sgn}(\sigma)$ signature of σ (parity of σ)

1 or −1

For $n \times n$ matrix $A = [a_{ij}]$, we define

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

Lemma 5.1 If $\sigma = \sigma_1 \cdots \sigma_k$

when $\sigma \in S_n$ $\underbrace{\sigma_1 \cdots \sigma_k}_{\text{transpositions}} \in S_n$

then

$\sigma^{-1} = \sigma_k \cdots \sigma_1$ in particular $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma^{-1})$.

Inverse function

in reverse order!!

proof

Note that $(i\,j)(i\,j) = id$.

Thus $(i\,j) = (i\,j)^{-1}$.

Hence

$$\underbrace{\sigma_1 \cdots \underbrace{\sigma_{k-1} \overbrace{\sigma_k \sigma_k}^{id} \sigma_{k-1}}_{id} \cdots \sigma_1}_{id} = id$$

$1 \cdots i \quad j \cdots n$

$1 \cdots i \quad j \cdots n$

$1 \cdots i \quad j \cdots n$

Hence $\sigma_k \cdots \sigma_1 = \sigma^{-1}$

$\boxtimes$

Theorem 5.2 For any $n \times n$-matrix A

$$\det(A) = \det({}^tA)$$

Lemma 5.3

$$I : S_n \to S_n;\ \sigma \mapsto \sigma^{-1}$$

is 1-1 onto mapping on S_n

proof To see that I is 1-1, suppose $I(\sigma) = I(\tau)$. $\quad \left(\sigma^{-1} = \tau^{-1}\right)$

Then $\quad \sigma = \underbrace{\tau\tau^{-1}}_{id}\sigma = \tau\underbrace{\sigma^{-1}\sigma}_{id} = \tau$. $\quad \left(\tau^{-1} = \sigma^{-1}\right)$

S_n is finite (actually S_n has exactly $n!$ elements)

So $I[S_n] \subseteq S_n$ and (since I is 1-1)

$I[S_n]$ has $n!$ elements. It follows that $I[S_n] = S_n$ ☒

proof of Thm 5.2 Note $\{(1, \sigma(1)), \dots (n, \sigma(n))\}$

$$= \{(\sigma^{-1}(1), 1), \dots, (\sigma^{-1}(n), n)\}$$

For $\sigma \in S_n$ $\quad \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$

$$= \operatorname{sgn}(\sigma)\, a_{\sigma^{-1}(1),1}\, a_{\sigma^{-1}(2),2} \cdots a_{\sigma^{-1}(n)n}$$

Lemma 5.1

$$= \operatorname{sgn}(\sigma^{-1})\, a_{\sigma^{-1}(1)1} \cdots a_{\sigma^{-1}(n),n}.$$

$$\begin{array}{ccccc} 1 & \cdots & \sigma^{-1}(1) & \cdots & n \\ \downarrow & & & & \downarrow \\ \sigma(1) & & 1 & & \sigma(n) \end{array}$$

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma^{-1})\, a_{\sigma^{-1}(1)1} \cdots a_{\sigma^{-1}(n)n}$$

$$\overset{\text{Lemma 5.3}}{=} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

$$= \det({}^tA).$$ ☒

Lemma 5.4

$$\det\Bigg(\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A}\Bigg) = a_{11} \det\Bigg(\overbrace{\begin{bmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \\ a_{n2} & \cdots & a_{nn} \end{bmatrix}}^{A_{11}}\Bigg)$$

$(n-1) \times (n-1)$ matrix

proof

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \overbrace{a_{1\sigma(1)}}^{0 \text{ if } \sigma(1) \neq 1} \cdots a_{n\sigma(n)}$$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma(1) = 1}} \operatorname{sgn}(\sigma) \overbrace{a_{1\sigma(1)}}^{a_{11}} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$$= a_{11} \overbrace{\Bigg(\sum_{\sigma : \{2, \dots n\} \xrightarrow[\text{onto}]{1-1} \{2, \dots n\}} \operatorname{sgn}(\sigma)\, a_{2\sigma(2)} \cdots a_{n\sigma(n)} \Bigg)}^{\det(A_{11})} = a_{11} \det(A_{11})$$ ☒

Theorem 5.2 For any $n\times n$-matrix A

$$\det(A) = \det({}^tA)$$

Cor 5.5

$$\det\left(\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\right) = a_{11}\det\left(\begin{bmatrix} a_{22} & a_{2n} \\ \vdots & \\ a_{n2} & \cdots a_{nn} \end{bmatrix}\right)$$

proof By Lemma 5.4 and Theorem 5.2. ☒

Theorem 5.6 (1) $\det(E) = 1$

(2) If $A = [a_1 \cdots a_n]$ Then

$$\det([a_1 \cdots \underbrace{ca_i}_{i} \cdots a_n]) = c\det(A)$$

(3) $$\det([a_1 \cdots \underbrace{a_i + b}_{} \cdots a_n]) = \det(A) + \det([a_1 \cdots \underbrace{b}_{i} \cdots a_n])$$

multi linearity of determinant

(3) $$\det([a_1 \cdots \underbrace{a_j}_{i} \; \underbrace{a_i}_{j} \cdots a_n]) = -\det([a_1 \cdots a_i \cdots a_j \cdots a_n])$$

Note the properties above characterize totally the determinant.

proof (1):

$$\det(E) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\delta_{1\sigma(1)} \cdots \delta_{n\sigma(n)} = \sum_{\sigma = id} \overbrace{\cdots\cdots}^{=1} = 1$$

$\boxed{\text{if } \sigma \neq id \text{ then } \delta_{k\sigma(k)} = 0 \text{ for } k \text{ with } \sigma(k) \neq k}$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

(2):

$$\det([a_1 \cdots \underbrace{ca_i}_{i} \cdots a_n]) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots ca_{i\sigma(i)} \cdots a_{n\sigma(n)}$$

$$= c\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{i\sigma(i)} \cdots a_{n\sigma(n)}$$

$$= c\det(A).$$

(3): Let $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$$\det([a_1 \cdots \underbrace{a_i + b}_{i} \cdots a_n])$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots \overbrace{(a_{i\sigma(i)} + b_{\sigma(i)})}^{i} \cdots a_{n\sigma(n)}$$

$$= \underbrace{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{i\sigma(i)} \cdots a_{n\sigma(n)}}_{= \det(A)} + \underbrace{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots \overset{i}{\widehat{b_{\sigma(i)}}} \cdots a_{n\sigma(n)}}_{= \det([a_1 \cdots \overset{i}{\check{b}} \cdots a_n])}$$

(4):

$\det([a_1 \cdots \underbrace{a_i}_{i} \; \underbrace{a_j}_{j} \cdots a_n])$

$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{i\sigma(i)} \cdots a_{j\sigma(j)} \cdots a_{n\sigma(n)}$

the transposition $(i\,j)$

$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(i\,j)(1)} \cdots a_{i\,\sigma(i\,j)(i)} \cdots a_{j\,\sigma(i\,j)(i)} \cdots a_{n\sigma(i\,j)(n)}$

$= \sum_{\sigma \in S_n} -\operatorname{sgn}(\sigma(i,j)) \cdots\cdots$

$= -\underbrace{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{i\sigma(i)} \cdots a_{j\sigma(j)} \cdots a_{n\sigma(n)}}_{\det(A)}$ ☒

Note for $k \notin \{i,j\}$

$\sigma(i\,j)(k) = \sigma(k)$

$\Phi : S_n \to S_n ;\ \sigma \mapsto \sigma(i\,j)$

is also 1-1 onto

Theorem 5.6 (1) $\det(E) = 1$

(2) If $A = [a_1 \cdots a_n]$ Then

$\det([a_1 \cdots \underbrace{c a_i}_{i} \cdots a_n]) = c \det(A)$

(3) $\det([a_1 \cdots \underbrace{a_i + b}_{} \cdots a_n]) = \det(A) + \det([a_1 \cdots \underbrace{b}_{i} \cdots a_n])$

multi linearity of determinant

(4) $\det([a_1 \cdots \underbrace{a_j}_{i} \; \underbrace{a_i}_{j} \cdots a_n]) = -\det([a_1 \cdots a_i \cdots a_j \cdots a_n])$

Note the properties above characterize totally the determinant.

proof (1):

$\det(E) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \delta_{1\sigma(1)} \cdots\cdots \delta_{n\sigma(n)} = \sum_{\sigma = id} \overbrace{\cdots\cdots}^{1} = 1$

$\boxed{\text{if } \sigma \neq id \text{ then } \delta_{k\sigma(k)} = 0 \text{ for } k \text{ with } \sigma(k) \neq k}$

Cor 5.7 Thm 5.6 with rows and columns switched.

proof By Thm 5.6 and Thm 5.2. ☒

[redacted]