Webpage of the course:

http:// fuchino.ddo.jp/kobe/index.html

Recap: A : n×n matrix

Thm 5.2 $\det(A) = \det({}^tA)$

Lemma 5.4

$$\det\left(\begin{bmatrix} a_{11} & \cdots\cdots \\ 0 & \\ \vdots & \boxed{A_{11}} \\ 0 & \end{bmatrix}\right) = a_{11}\det(A_{11})$$

$\uparrow$ $(n-1)\times(n-1)$

Cor 5.5

$$\det\left(\begin{bmatrix} a_{11} & 0 \cdots\cdots 0 \\ \vdots & \boxed{A_{11}} \end{bmatrix}\right) = a_{11}\det(A_{11})$$

Proof

$$\det\left(\begin{bmatrix} a_{11} & 0 \cdots 0 \\ \vdots & [A_{11}] \end{bmatrix}\right) \overset{\text{Thm 5.2}}{=} \det\left({}^t\begin{bmatrix} & \\ & 〃 \\ & \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_{11} & \cdots\cdots \\ 0 & \\ \vdots & \boxed{{}^tA_{11}} \\ 0 & \end{bmatrix}\right)$$ □

$a_{11}\det(A_{11})$

$\parallel \leftarrow$ Thm 5.2

$a_{11}\det({}^tA_{11}) \swarrow$ Lemma 5.4

$\parallel$

Thm 5.6 (1) $\det(E) = 1$

(2) $\det[a_1 \cdots c\underset{i}{a_i} \cdots a_n] = c\det[a_1 \cdots \underset{i}{a_i} \cdots a_n]$

(3) $\det[a_1 \cdots \underset{i}{a+b} \cdots a_n] = \det[a_1 \cdots \underset{i}{a} \cdots a_n] + \det[a_1 \cdots \underset{i}{b} \cdots a_n]$

(4) $\det[a_1, \underset{i}{a_i} \curvearrowright \underset{j}{a_j} \cdots a_n] = -\det[a_1 \cdots a_j \cdots a_i \cdots a_m]$

Cor 5.7
The "col" version of Thm 5.6

Lemma 6.1 Let

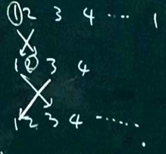
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & k & k+1 & \cdots & n \\ & & & & & \downarrow & & \downarrow \\ k & 1 & 2 & 3 \cdots & k-1 & k+1 & \cdots\cdots & n \end{pmatrix}$$

$\operatorname{sgn}\sigma = (-1)^{k+1}$

proof $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & \cdots & k-1 & k & k+1 & \cdots & n \\ 2 & 3 & 4 & \cdots & k & 1 & k+1 & \cdots & n \end{pmatrix}$

$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$

$$\sigma^{-1} = \underbrace{(1\,k) \cdots (1\,4)(1\,3)(1\,2)}_{k-1}$$



$\Rightarrow \operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1}) = (-1)^{k-1} = (-1)^{k+1}$.

⊠

Theorem 6.2

(1) If

$$A = \begin{array}{c} \\ \\ i \\ \\ \\ \end{array}\begin{bmatrix} 0 & \\ \vdots & \\ a_{i1} & \text{———} \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}$$

then $\det(A) = (-1)^{i+1} A_{i1}$ for $1 \le i \le n$

(2)

$$A = \begin{array}{c} \\ \\ i \\ \\ \\ \end{array}\begin{bmatrix} & \overset{j}{0} & \\ & \vdots & \\ \text{——} & a_{ij} & \text{——} \\ & \vdots & \\ & 0 & \end{bmatrix}$$

Then $\det(A) = (-1)^{i+j} A_{ij}$ for $1 \le i, j \le n$

Notation For n×n matrix $A = [a_{ij}]$ and $1 \le i, j \le n$

A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by eliminating ith row and jth column from A. That is,

$$A_{ij} = [a^*_{kl}] \text{ where } a^*_{kl} = a_{k'l'} \text{ with } k' = \begin{cases} k & \text{if } k < i \\ k-1 & \text{if } k > i \end{cases} \quad l' = \begin{cases} l & \text{if } l < j \\ l-1 & \text{if } l \ge j \end{cases}$$

Proof of Thm 6.2:

(1) If $i=1$ Then the assertion is just that of Lemma 5.4. Suppose $i>1$

Let $A=\begin{bmatrix} b_1\\ \vdots \\ b_n\end{bmatrix}$ $A'=\begin{bmatrix} b_{\sigma(1)}\\ \vdots\\ b_{\sigma(n)}\end{bmatrix} = \left[\begin{array}{c|c} a_{i1} & \cdots\cdots \\ \hline 0 & \\ \vdots & A_{i1}\\ 0 & \end{array}\right]$

where $\sigma = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ i & 1 & 2 \cdots & i-1 & i+1 & \cdots & n\end{pmatrix}$

By Lemma 6.1 $\mathrm{sgn}(\sigma)=(-1)^{i+1}$

By Cor 5.7 (4)

$$\det A = (-1)^{i+1}\det A' \overset{\text{Lemma 5.4}}{=} (-1)^{i+1} a_{i1}\det(A_{i1}).$$

(2): If $j=1$ Then this is just the case (1)

Suppose $j>1$

Let $A=[a_1 \cdots a_n]$

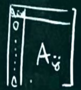
$\mu = \begin{pmatrix} 1 & 2 & \cdots & j & j+1 & \cdots & n\\ j & 1 & \cdots & j-1 & j+1 & \cdots & n\end{pmatrix}$

By Lemma 6.1 $\mathrm{sgn}(\mu) = (-1)^{j+1}$

$A^{\dagger}=$

$A^\dagger = [a_{\mu(1)}\, a_{\mu(2)} \cdots a_{\mu(n)}]$

$A^{\dagger'}$



$$\det(A) = (-1)^{j+1}\det(A^\dagger) \underset{\uparrow\,(1)}{=} (-1)^{j+1}(-1)^{i+1}\det(A^{\dagger'})$$

$$= \underbrace{(-1)^{i+j+2}}_{=(-1)^{i+j}} a_{ij}\det(A_{ij})$$

☒

Thm 5.6 (1) $\det(E)=1$

(2) $\det[a_1 \cdots c\underset{i}{a_i} \cdots a_n] = c\det[a_1 \cdots \underset{i}{a_i} \cdots a_n]$

(3) $\det[a_1 \cdots \underset{i}{a+b} \cdots a_n] = \det[a_1 \cdots \underset{i}{a} \cdots a_n] + \det[a_1 \cdots \underset{i}{b} \cdots a_n]$

(4) $\det[a_1 \cdots \underset{i}{a_j} \curvearrowright \underset{j}{a_i} \cdots a_n] = -\det[a_1 \cdots a_i \cdots a_j \cdots a_n]$

Cor 5.7

The "row" version of Thm 5.6

Theorem 6.3 Let A be an $n\times n$-matrix with $A=[a_{ij}]$ Then for fixed $1 \leqslant j \leqslant n$

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

proof Let $A=[a_1 \cdots a_n]$

$$\det(A) = \det\left([a_1 \cdots \overbrace{\begin{bmatrix}a_{1j}\\0\\\vdots\\0\end{bmatrix}+\begin{bmatrix}0\\a_{2j}\\0\\\vdots\\0\end{bmatrix}+\cdots+\begin{bmatrix}0\\\vdots\\0\\a_{nj}\end{bmatrix}}^{j} \quad a_n]\right)$$

$$\overset{\text{Thm 5.6 (3)}}{=} \det\left([a_1 \cdots \overset{j}{\begin{bmatrix}a_{1j}\\0\\\vdots\\0\end{bmatrix}} \cdots a_n]\right) + \det\left([a_1 \cdots \overset{j}{\begin{bmatrix}0\\a_{2j}\\0\\\vdots\\0\end{bmatrix}} \cdots a_n]\right) + \cdots + \det\left([a_1 \cdots \overset{j}{\begin{bmatrix}0\\\vdots\\0\\a_{nj}\end{bmatrix}} \cdots]\right)$$

$\| \leftarrow$ Thm 6.2 $(-1)^{1+j} a_{1j} \det A_{1j}$

$\| \leftarrow$ Thm 6.2 $(-1)^{2+j} a_{2j} \det A_{2j}$

$\|$ $(-1)^{n+j} a_{nj} \det A_{nj}$

☒

Lemma 6.5. If $A=[a_1 .. a_n]$ and $a_i = a_j$ for $i\neq j$ Then $\det(A)=0$

proof

$\det(A) \overset{\text{by Th. 5.6 (4)}}{=} -\det[a_1 \cdots a_j \curvearrowright a_i \cdots a_n]$

$= -\det(A)$

$\Rightarrow \det(A) = 0$ ☒

Lemma 6.4. For an $n\times n$ matrix A with $A=[a_{ij}]$

Let $\hat{A} = [a^*_{ij}]$ where

$a^*_{ij} = (-1)^{i+j}\det(A_{ji})$ cofactors of A

Then $A\widetilde{A} = \hat{A}A = \det(A)E$

proof

Let $A\hat{A} = [c_{ij}]$ Then

$c_{ij} = \sum_{k=1}^{n} a_{ik} a^*_{kj}$ If $i=j$

$c_{ii} = \sum_{k=1}^{n} a_{ik} (-1)^{i+k} \det(A_{ik}) \overset{\text{Theorem 6.3}}{=} \det(A)$

If $i\neq j$, letting $A=\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}$, let $B = j\rightarrow\begin{bmatrix}b_1\\ \vdots\\ b_i\\ \vdots\\ b_n\end{bmatrix}$

$c_{ij} = \det(B) \overset{\text{Lemma 6.5}}{=} 0$ | Similarly for $\hat{A}A$ ☒

Thm 6.5 An $n \times n$-matrix A is invertible iff $\det(A) \neq 0$ and If A is invertible Then

$$A^{-1} = \frac{1}{\det(A)} \tilde{A}$$

proof If $\det(A) \neq 0$ then $\frac{1}{\det(A)} \tilde{A}$ is the the inverse of A by Lemma 6.4. If A is invertible then

$$\underset{\substack{\uparrow \\ \text{Theorem 5.6 (1)}}}{1 =} \det(E) = \det(AA^{-1}) = \det(A)\det(A^{-1}) \Rightarrow \det(A) \neq 0$$

Thm 5.6 (1) $\det(E) = 1$

(2) $\det[a_1 \cdots \underset{\smile}{ca_i} \cdots a_n] = c \det[a_1 \cdots \underset{\smile}{a_i} \cdots a_n]$

(3) $\det[a_1 \cdots \underset{\smile}{a+b} \cdots a_n] = \det[a_1 \cdots \underset{i}{a} \cdots a_n] + \det[a_1 \cdots \underset{i}{b} \cdots a_n]$

(4) $\det[a_1 \cdots \underset{i}{a_j} \curvearrowleft \underset{j}{a_i} \cdots a_n] = -\det[a_1 \cdots a_i \cdots a_j \cdots a_m]$

Cor 5.7 The "row" version of Thm 5.6

Thm 6.6 $(n \times n)$-matrix A is invertible if there is an $(n \times n)$-matrix B s.t.

$AB = E$ (we need not to check $BA = E$!!)

In this case we have $B = A^{-1}$! $\quad \det(A)\det(B) \overset{\shortparallel}{} $

Proof If $AB = E$ then $\quad \det(AB) = \det(E) = 1$

$$\underset{(1)}{\Rightarrow \det(A) \neq 0} \underset{\substack{\uparrow \\ \text{Thm 6.5}}}{\Rightarrow} A \text{ is invertible} \quad (↗)$$

(↗) Since $AB = E$

$$\underbrace{A^{-1}A}_{\substack{\shortparallel \\ E}} B = A^{-1}E = \underline{\underline{A^{-1}}}$$

$$\shortparallel$$

$$\underline{\underline{B}}$$

□

Lemma 6.5 If $A = [a_1 \ldots a_n]$ and $a_i = a_j$ for $i \neq j$ Then $\det(A) = 0$

proof

$$\det(A) \overset{\text{by Th. 5.6 (4)}}{=} -\det[a_1 \ldots \overset{\curvearrowleft}{a_j \, a_i} \ldots a_n] = -\det(A)$$

$\Rightarrow \det(A) = 0$ ⊠

Lemma 6.4. For an $n \times n$ matrix A with $A = [a_{ij}]$

Let $\hat{A} = [a^*_{ij}]$ where $a^*_{ij} = (-1)^{i+j} \det(A_{ji})$ cofactors of A

Then $A\tilde{\hat{A}} - \hat{A}A = \det(A)E$

proof

Let $A\hat{A} = [c_{ij}]$ Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} a^*_{kj} \quad \text{If } i = j$$

$$c_{ii} = \sum_{k=1}^{n} a_{ik} (-1)^{i+k} \det(A_{ik}) \overset{\swarrow \text{Theorem 6.3}}{=} \det(A)$$

If $i \neq j$, letting $A = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$, let $B = {}_{j \rightarrow} \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ b_n \end{bmatrix}$

$$c_{ij} = \det(B) \overset{\swarrow \text{Lemma 6.5}}{=} 0$$

Similarly for $\hat{A}A$ ⊠