Set theory : "Theory of everything" in mathematics.

All existing mathematical theories can be formulated in the framework of Set Theory!

What for :

(1) We can formalize all the math. (theoretically !!)

arguments in a strictly formal language, so that the correctness of the arguments can be checked by a computer!

(2) Possibility of automatical theorem proving.

(3) In any case we can formulate the whole mathematics in a uniform framework.

(4) We can treat the fundational question like (in)consistency and completeness of mathematics in a uniform framework.

(5) The Incompleteness Theorems imply that the mathematics (= set theory) is not complete, in the sense that

There are mathematical assertions which can not be proved nor disproved from the usual axioms of mathematics

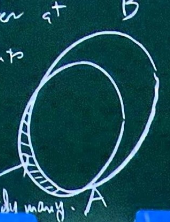
Examples The statement is independent from the axiom system of set theory

— The continuum hypothesis.
(There is no size of infinity between the size of $\mathbb{N}$ and the size of $\mathbb{R}$)

— For $A, B \subseteq \mathbb{N}$

$A \sim_{fin} B \iff$ if A and B differ at finitely many points

$A \subseteq^* B \iff B \setminus A$ is finite



$\mathcal{P}(\mathbb{N})/fin$ the equivalence class of $\sim_{fin}$ with $\subseteq^*$

"$Aut_0(\mathcal{P}(\mathbb{N})/fin)$ is simple" is independent!

automorphism group of $\mathcal{P}(\mathbb{N})/fin$ wrt $\subseteq^*$

⋮

Presentation Room in 3号館 412? in 渕野グループ

Naive axiomatic set theory

A theory with predicate symbols "=" "$\in$". We read $a=b$ "a and b are equal", $a \in b$ a is an element of b.

Here all the objects of the theory are "sets".

Axiom of Extensionality

For a and b $a=b$ if and only if (for any c $c \in a \overset{\text{if and only}}{\Longleftrightarrow} c \in b$)

http://fuchino.ddo.jp/kobe/index.html#math-logic-2019
(lecture notes of 2018)
http://fuchino.ddo.jp/katowice/

Axiom of empty set

There is (a set) a s.t. for any c we have $c \notin a$

$\uparrow \text{not}(c \in a)$

The set a above is unique by the Axiom of Extensionality. We call a an empty set and denote it by $\emptyset$.

Pairing Axiom

For a, b there is c s.t. for any d

$d \in c \Leftrightarrow d = a$ or $d = b$

c is determined uniquely from a and b (by Axiom of Extensionality). We denote $c = \{a, b\}$. If $a = b$ we write $c = \{a\}$ (or $c = \{b\}$) read singleton a

Combining these axioms we obtain

$\emptyset$, $\{\emptyset\}$

Lemma 1 $\emptyset \neq \{\emptyset\}$

proof $\emptyset$ does not have any element by definition. But $\emptyset \in \{\emptyset\}$. By Extensionality, it follows that $\emptyset \neq \{\emptyset\}$ $\boxtimes$

We can also construct $\{\emptyset, \{\emptyset\}\}$

$\{\{\emptyset, \{\emptyset\}\}\}$ $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ ...

Ordered Pairs

Note that $\{a, b\} = \{b, a\}$

For a, b we define

$\langle a, b \rangle = \{\{a\}, \{a, b\}\}$

Lemma 2 For any a, b, a', b'

$\langle a, b \rangle = \langle a', b' \rangle$ iff $a = a'$ and $b = b'$

Proof Exercise! $\boxtimes$

Axiom of Union

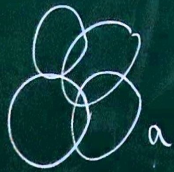
For any a there is u s.t.

for any c $c \in u \Leftrightarrow$ there is $b \in a$ s.t. $c \in b$

---

a set a is also a family of sets

If $a = \{b, d\}$ then



$c \in u \Leftrightarrow c \in b$ or $c \in d$

In this case we denote $u = b \cup d$

We also write $u = \bigcup a$

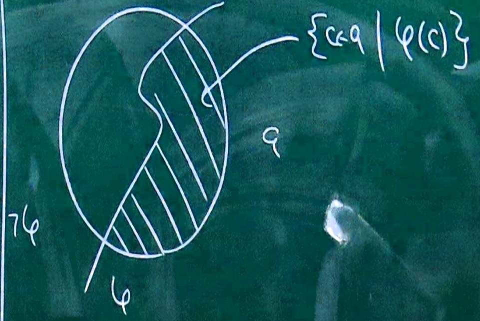
Axiom of Separation

If $\varphi(\cdot)$ is some property expressed by using $=$ and $\in$ then

for any a there is b s.t.

for any c $c \in b \Leftrightarrow c \in a$ and $\varphi(c)$

We write $b = \{c \in a \,;\, \varphi(c)\}$



Power set Axiom

Axiom of Infinity

Zermel Axiom of Set Theory ...