Recap:

Axioms of Set Theory

(In a naive axiomatic setting)

- Axiom of Extensionality
- Axiom of Empty set. ϕ
- Pairing Axiom

http://fuchino.ddo.jp/kobe/

$\{a, b\}$ $\{a\}$

$\langle a, b\rangle = \{\{a, b\}, \{b\}\}$

Exercise For any a, b, a', b'

$\langle a, b\rangle = \langle a', b'\rangle \Leftrightarrow a = a'$ and $b = b'$

Note that, if $a = b$ then $\langle a, a\rangle = \{\{a\}, \{a\}\} = \{\{a\}\}$

Lemma 1.1 $\phi \neq \{\phi\}$

Axiom of Union $\bigcup a$ if $a = \{c, d\}$ Then we write $\bigcup a = c \cup d$.

Axiom of Separation: If $\varphi(\cdot)$ is some property formulated using "$=$", "$\in$" and a there is b s.t.

for any c $c \in b \Leftrightarrow c \in a$ and $\varphi(c)$

We write $b = \{c \in a \mid \varphi(c)\}$

— φ may contain some other sets say $a_0, \ldots a_{n-1}$ as parameters

i.e. $\varphi(\cdot)$ may be $\psi(\cdot, a_0, \ldots a_{n-1})$

for some other property ψ

Lemma 2.1 For any property $\varphi(c)$ as above there is a set a s.t.

(0) if there is no c with $\varphi(c)$ then $a = \phi$.

(1) if there is (at least one) c with $\varphi(c)$ then, for every d,

$d \in a \Leftrightarrow d \in b$ for any b s.t. $\varphi(b)$

Intuitively $a = \bigcap \underbrace{\{b \mid \varphi(b)\}}_{\text{this is not a set in general.}}$

Proof (0) is o.k.

Assume that there's some c with $\varphi(c)$.

(Idea: a if it exists should be a "subset" of c!)

$$a = \underbrace{\{d \in c \mid \overbrace{d \in b \text{ for all } b \text{ s.t. } \varphi(b)}\}}_{\text{this is a set by Axiom of Separation}}$$

and this a is as desired! $\boxtimes$

For a, b, a is said to be a subset of b if for any $c \in a$, $c \in b$ holds.

If a is a subset of b then we denote this by $a \subseteq b$ ($a \subseteqq b$). Axiom of Extensionality implies

$a \subseteq b$ and $b \subseteq a \Rightarrow a = b$

Axiom of Infinity There is a with the following property (a) $\emptyset \in a$, (b) If $b \in a$ then $b \cup \{b\} \in a$

$$\underbrace{(= \bigcup\{b, \{b\}\})}$$

For the set a as in the Axiom

$0 := \emptyset \in a$

$1 := \emptyset \cup \{\emptyset\} = \{\emptyset\} \in a$

$\{\emptyset\} \cup \{\{\emptyset\}\} \in a$ $\qquad 0, 1, 2, \ldots \in a$

$2 := \{\emptyset, \{\emptyset\}\}$

$\{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} \in a$

$3 := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ ....

a in the Axiom of Infinity is not unique.

Let φ be the property saying that $\varphi(a) \Leftrightarrow a$ satisfies (a) and (b)

Define

$$\mathbb{N} = \bigcap \{a \mid \varphi(a)\}$$

There is the set $\mathbb{N}$ by Lemma 2.1

$\mathbb{N}$ satisfies (a) and (b)

Lemma ?.2 $\mathbb{N}$ satisfies (a) & (b) in the formulation of the Axiom of infinity

Proof Note that

$d \in \mathbb{N} \iff d \in b$ for any b with $\varphi(b)$

$\emptyset \in \mathbb{N}$, since $\emptyset \in b$ with $\varphi(b)$

Suppose that $d \in \mathbb{N}$ then $d \in b$ for all b with $\varphi(b)$

It follows that $d \cup \{d\} \in b$ for all b with $\varphi(b)$

Thus $d \cup \{d\} \in \mathbb{N}$. This shows that $\mathbb{N}$ satisfies (b). $\boxtimes$

(Almost ?) everything in classical analysis (everything taught in Calculus I II III etc 微積分) can be reformulated in the axiom system consisting of Axioms introduced before.

Powerset Axiom: For any a there is b s.t. for any c, $c \in b \iff c \subseteq a$

Such b is called the powerset of a and denoted by

$b = \mathcal{P}(a)$

The axiom system consisting of all the axioms introduced before is called the Zermelo set theory and denoted by "Z"

Zermelo introduced a system (not quite identical with those here but more or less the same) in his 1908 paper.

The treatment of the notion of functions in Z (and in extensions of Z)

For A, B let

$A \times B = \{\langle a,b\rangle \mid a \in A,\ b \in B\}$

i.e. let $A \times B$ be the set with the property that $c \in A \times B \iff$ there are $a \in A$ and $b \in B$ s.t. $c = \langle a,b\rangle$

Lemma ?.3 $A \times B$ exists.

Proof Assuming that $A \times B$ exists

$c \in A \times B \iff$ there are $a \in A$ $b \in B$ with $c = \langle a,b\rangle = \{\{a,b\},\{b\}\}$

$\left.\begin{array}{l}\{a,b\} \\ \{b\}\end{array}\right\} \subseteq A \cup B \qquad \begin{array}{l}\{a,b\} \\ \{b\}\end{array} \in \mathcal{P}(A \cup B)$

$\{\{a,b\},\{b\}\} \subseteq \mathcal{P}(A \cup B)$

$\langle a,b\rangle \in \mathcal{P}(\mathcal{P}(A \cup B))$

$A \times B = \underbrace{\{c \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid c = \langle a,b\rangle \text{ for some } a \in A \text{ and } b \in B\}}$

exists!

For any A, B a function from A to B is a subset F of $A \times B$ s.t. for any $a \in A$ there is a unique $b \in B$ s.t. $\langle a, b \rangle \in F$

If F is a function from A to B we denote this by $F : A \to B$

For any $a \in A$ the unique $b \in B$ with $\langle a, b \rangle \in F$ is denoted by $F(a)$.

Lemma 2.4 For A, B there is the set ${}^A B = \{F ; F : A \to B\}$

Proof $\boxed{\text{Assuming } {}^A B \text{ exists}}$

$\boxed{F \in {}^A B \Rightarrow F \subseteq A \times B \Leftrightarrow F \in \mathcal{P}(A \times B)}$

${}^A B = \left\{ F \in \mathcal{P}(A \times B) \;\middle|\; \begin{array}{l} \text{For any } a \in A \text{ there is a unique } b \in B \\ \text{s.t. } \langle a, b \rangle \in F \end{array} \right\}$

This exists by Lemma 2.3

exists. ∎

Using these we can define $\mathbb{R}$ ${}^{\mathbb{R}}\mathbb{R}$ …

… in Z.