Lemma 3.? (1) For any $n, m \in \mathbb{N}$ exactly one of $n \in m$ or $n = m$ or $m \in n$ holds

(2) For all $n, m \in \mathbb{N}$

$$n \in m \iff n \subsetneq m$$

(3) For any $n, m \in \mathbb{N}$ we have either $n \subsetneq m$ or $n = m$ or $m \subsetneq n$

(4) $\in$ (or equivalently $\subsetneq$) is a linear ordering on $\mathbb{N}$.

Treatment of binary relations in Z

For a set A a relation on A is a subset of $A^2 = A \times A$

Remark A function $f : A \to A$ is a binary relation on A.

If $R \subseteq A^2$ and $\langle a, b\rangle \in R$ for $a, b \in A$. We write this also as aRb.

$\sqsubset \subseteq A^2$ is a (partial) ordering on A if

① $a \not\sqsubset a$ (i.e. $\langle a, a\rangle \notin \sqsubset$) for all $a \in A$

② for all a, b, c, $a \sqsubset b$ and $b \sqsubset c$ imply $a \sqsubset c$ (the relation is transitive)

$\sqsubset \subseteq A^2$ is a linear ordering if

$\sqsubset$ satisfies $\underbrace{① ②}_{\text{partial ordering}}$ and

③ for all $a, b \in A$ one of $a \sqsubset b$, $a = b$, $b \sqsubset a$ holds.

Remark ① If $\sqsubset$ is a partial ordering then $a \sqsubset b$ implies $b \not\sqsubset a$ [if $a \sqsubset b$, $b \sqsubset a$ Then ② implies $a \sqsubset a$ which contradicts ①]

Thus the conditions $a \sqsubset b$, $a = b$, $b \sqsubset a$ are exclusive.

② Let $\sqsubset = \{\langle a, b\rangle \in \mathbb{N}^2 \mid a \in b\}$ $= \in\restriction\mathbb{N}$. (4) claims that $\sqsubset$

(which is $= \{\langle a, b\rangle \in \mathbb{N}^2 \mid a \subsetneq b\}$) $= \subsetneq\restriction\mathbb{N}$

is a linear ordering on $\mathbb{N}$

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proof (1): done last time.

(2): Suppose $n, m \in \mathbb{N}$.

If $n \in m$ then $n \subseteq m$ by Lemma 3.1, (2).

$n \neq m$ by Lemma 3.1, (3) Thus we have $n \subsetneq m$

Remark + Exercise For a TFAE:

(a) a is transitive

(b) For any b, $b \in a \Rightarrow b \subseteq a$

If $n \subsetneq m$ then $n \neq m$ and $m \notin n$ [if $m \in n$ then $m \in n \subsetneq m$. Hence $m \in m$ a contradiction to Lemma 3.1,(3),

By (1), it follows that $n \in m$.

(3): clear by (1) and (2)

(4): $\in\restriction\mathbb{N} = \subsetneq\restriction\mathbb{N}$ by (2)

$\in\restriction\mathbb{N}$ is transitive since $\in\restriction\mathbb{N} = \subsetneq\restriction\mathbb{N}$

$a \notin a$ by Lemma 3.1,(3) for all $a \in \mathbb{N}$

and the linearity ③ is just (1). $\boxtimes$

Lemma 4.1 ($s : \mathbb{N} \to \mathbb{N};\ n \mapsto n \cup \{n\}$)

For all $n \in \mathbb{N}$: $s(n)$ is a successor of n wrt $\in\restriction\mathbb{N}$

(i.e. $n \in s(n) = n \cup \{n\}$ and for any m with $n \in m$ Then $s(n) \in m$ or $s(n) = m$)

proof $n \in s(n) = n \cup \{n\}$ is clear

Suppose $n \in m$. Then We have either $s(n) \in m$ or $s(n) = m$ or $m \in s(n)$ by Lemma 3.2 (1).

$m \notin s(n)$ [If $m \in s(n)$ · $m \in n$ or $m = n$

$\Downarrow$ $n \in n$ ↯ $\Downarrow$ $m \in m$ ↯

So we have $s(n) \in m$ or $s(n) = m$ as desired !. $\boxtimes$

Notation $\in\restriction\mathbb{N}$ is also denoted by $<$

$s(n)$ by $n+1$

Lemma 4.2 There is a unique function $add : \mathbb{N}^2 \to \mathbb{N}$ with

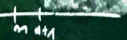
① $add(\langle n, 0\rangle) = n$

② $add(\langle n, m+1\rangle) = add(\langle n, m\rangle) + 1$

proof we first prove the uniqueness. Suppose that $add : \mathbb{N}^2 \to \mathbb{N}$ is as above

$n+1$ is The smallest element of $\mathbb{N}$ bigger Than n

i.e. for any m with $n < m$ either $n+1 < m$ or $n+1 = m$



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and $add^* : \mathbb{N}^2 \to \mathbb{N}$ with

①* $add^*(\langle n, 0\rangle) = n$

②* $add^*(\langle n, m+1\rangle) = add^*(\langle n, m\rangle) + 1$

We show that, for all $n \in \mathbb{N}$

$(*)_m$ for all $m \in \mathbb{N}$ $add(\langle n, m\rangle) = add^*(\langle n, m\rangle)$

We show ⓐ $(*)_0$ and ⓑ $(*)_m \to (*)_{m+1}$

For ⓐ:

$$add(\langle n, 0\rangle) \underset{\substack{\uparrow \\ ①}}{=} n \underset{\substack{\uparrow \\ ①^*}}{=} add^*(\langle n, 0\rangle)$$

For ①: assume $(*)_m$

$$\mathrm{add}(\langle n, m+1\rangle) \underset{\uparrow ②}{=} \mathrm{add}(\langle n, m\rangle) + 1$$

$$\underset{\uparrow (*)_m}{=} \mathrm{add}^*(\langle n, m\rangle) + 1 \underset{\uparrow ②^*}{=} \mathrm{add}^*(\langle n, m+1\rangle).$$

To show the existence we show that for all $m \in \mathbb{N}$ there is a unique function $\mathrm{add}_m : \mathbb{N} \times m+1 \to \mathbb{N}$ with ① ②

[Note $m+1 = \{n \in \mathbb{N} \mid n \leq m\}$ by Lemma 4.3 (3)] and that

⊛ for $m < m'$ we have $\mathrm{add}_m \subseteq \mathrm{add}_{m'}$

The uniqueness is proved similarly to the uniqueness of add ②

<u>Lemma 4.3</u> (1) For any $n \in \mathbb{N}$ we have $n \subseteq \mathbb{N}$.

(2) If $n \in \mathbb{N}$ then $n = \{m \in \mathbb{N} \mid m < n\}$

(3) $n+1 = \{m \in \mathbb{N} \mid m \leq n\}$

<u>proof</u> (1): Let $X = \{n \in \mathbb{N} \mid n \subseteq \mathbb{N}\}$

Then $\phi \in X$. if $n \in X$ then $n \in \mathbb{N}$ (and $n \subseteq \mathbb{N}$)

So $n+1 = n \cup \{n\} \in \mathbb{N}$ and $n \cup \{n\} \subseteq \mathbb{N}$

Thus $n+1 \in X$. Thus $X = \mathbb{N}$.

(2): If $m \in n$ then $m \in \mathbb{N}$. Hence $m \in \{m \in \mathbb{N} \mid m \underset{\in}{<} n\}$

If $m \in \{m \in \mathbb{N} \mid m \underset{\in}{<} n\}$ then trivially $m \in n$. ☒

(3): $n+1 = \{m \in \mathbb{N} \mid m < n+1\}$: by (2)

$= \{m \in \mathbb{N} \mid m < n \vee m = n\}$. $(n+1 = n \cup \{n\})$

① add_0 can be defined as

$$\{\langle\langle n, 0\rangle, n\rangle \in \mathbb{N}^2 \times \mathbb{N} \mid n \in \mathbb{N}\} = \{m \in \mathbb{N}^2 \times \mathbb{N} \mid \text{there is } n \in \mathbb{N}\ m = \langle\langle n, 0\rangle, n\rangle\}.$$

Suppose that add_m has been defined then let

$$\mathrm{add}_{m+1} = \mathrm{add}_m \cup \{\langle\langle n, m+1\rangle, \mathrm{add}_m\langle n, m\rangle + 1\rangle \in \mathbb{N}^2 \times \mathbb{N} \mid n \in \mathbb{N}\}$$

Since $\mathrm{add}_{m+1} \supseteq \mathrm{add}_m$

⊛ is also satisfied.

Finally we define

$$\mathrm{add} = \bigcup_{m \in \mathbb{N}} \mathrm{add}_m = (*)$$

$(*) =$

$$\{\langle\langle n, m\rangle, \ell\rangle \in \mathbb{N}^2 \times \mathbb{N} \mid \text{there is some } m^* \in \mathbb{N} \text{ s.t. } \langle\langle n, m\rangle, \ell\rangle \in \mathrm{add}_{m^*}\}.$$

This add is as desired. ☒

Similarly we can show the unique existence $\mathrm{mult} : \mathbb{N}^2 \to \mathbb{N}$ with

$\mathrm{mult}(n, 0) = 0$

$\mathrm{mult}(n, m+1) = \mathrm{add}(\mathrm{mult}(n, m), n)$. ☒