Lemma 3.1 (1) For $X \subseteq \mathbb{N}$, if

ⓐ $\emptyset \in X$

ⓑ For any $n \in X$ we have $n+1 \in X$

then $X = \mathbb{N}$

Lemma 4.3 (1) For any $n \in \mathbb{N}$ we have $n \subseteq \mathbb{N}$ ($\mathbb{N}$ is transitive (wrt $\in$))

(2) If $n \in \mathbb{N}$ $\quad n = \{m \in \mathbb{N} \mid m \in n\}$ $\quad <$

(3) $n+1 = \{m \in \mathbb{N} \mid m \leq n\}$

Lemma 5.1 (cumulative induction principle)

For $X \subseteq \mathbb{N}$ if

ⓐ $\emptyset \in X$

ⓑ' If $n \in X$ and $n \subseteq X$ then $n+1 \in X$

Then $X = \mathbb{N}$.

ⓑ $\Rightarrow$ ⓑ'

Remark The principle in Lemma 5.1 implies the principle in Lemma 3.1

Proof Let $X_0 = \{n \in X \mid n \subseteq X\}$. We have $\underbrace{X_0 \subseteq X}$

So it is enough to show that $X_0 = \mathbb{N}$

For this we check X_0 satisfies ⓐ ⓑ.

$\emptyset$ is a subset of any set. Hence

$\underbrace{\emptyset \in X}_{\text{by } X \models (a)}$ and $\emptyset \subseteq X$. Thus $\emptyset \in X_0$.

To show $X_0 \models$ ⓑ Suppose that $\underbrace{n \in X_0}$

Then, by def. of X_0, $\underbrace{n \subseteq X}_{n \in X_0 \subseteq X}$

Thus, by $X \models (b)'$, we have $n+1 \in X$ $\quad$ Hence $n+1 = n \cup \{n\} \subseteq X$

It follows that $\underbrace{n+1 \in X_0}$

Thus, by Lemma 3.1 (1), $X_0 = \mathbb{N}$

and hence also $X = \mathbb{N}$. $\boxtimes$

Lemma 5.2 ($<$ on $\mathbb{N}$ is a well-ordering)

For any non-empty $X \subseteq \mathbb{N}$ there is $m \in X$

s.t. $\underbrace{m \leq n}_{m = n \text{ or } m \in n}$ for all $n \in X$

A linear ordering R on a set X is said to be a well ordering if each non empty $Y \subseteq X$ has the minimal element wrt R

Proof Suppose that $X \subseteq \mathbb{N}$ does not have the minimal element. We show that $X = \emptyset$.

Let $Y = \mathbb{N} \setminus X$. It is enough to show $Y = \mathbb{N}$. We show $Y \models$ ⓐ, ⓑ'

$\emptyset \in Y$ [Otherwise $\emptyset \in X$ and $\emptyset$ would be the minimal element of X.]

To show $Y \models$ ⓑ' suppose that $n \in Y$ and $n \subseteq Y$, we have to show that $n+1 \in Y$. Suppose otherwise i.e. $n+1 \in X$ then $n+1$ is first element in X i.e. $n+1$ is the minimal element of X. A contradiction!

$X \setminus Y = \{x \in X \mid x \notin Y\}$

$\uparrow$

"the complement of Y in X"

"X set minus Y"

Thus $n+1 \in Y$ and $Y \models$ ⓑ'. ⊠

In the last lecture the functions add: $\mathbb{N}^2 \to \mathbb{N}$ mult: $\mathbb{N}^2 \to \mathbb{N}$. If we write add$(\langle n,m\rangle) =: n+m$ mult$(\langle n,m\rangle) =: n \cdot m$. Then $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$ satisfies all the properties of natural numbers e.g. formulated by Dedekind a Peano! $0+1 = \{\emptyset\}$ (Exercise!)

Thm 5.5 (Cantor's Theorem) (in Z)

For any set X there is no surjection $f: X \to \mathcal{P}(X)$

Proof Suppose $f: X \to \mathcal{P}(X)$. We show that f is not a surjection.

$$a = \{x \in X \mid x \notin f(x)\} \in \mathcal{P}(X)$$

We show that there is no $x \in X$ with $f(x) = a$: Suppose otherwise and let $x^* \in X$ be s.t. $f(x^*) = a$

$f: X \to Y$ is a surjection ($f \subseteq X \times Y$) if for all $y \in Y$ there is $x \in X$ s.t. $f(x) = y$

We should have $x^* \in a \ (= f(x^*))$ or $x^* \notin a \ (= f(x^*))$

If $x^* \in a$, then $x^* \in f(x^*)$. By def of a it follows that $x^* \notin a$. A contradiction.

If $x^* \notin a$ then $x^* \notin f(x^*)$. By def. of a $x^* \in a$. A contradiction. Thus there is no such $x^* \in X$! ⊠

Thm 5.4 (Russell's Paradox)

There is no a with $a = \{x \mid x \notin x\}$

Proof Assume there were $a = \{x \mid x \notin x\}$.

Then we have $a \in a$ or $a \notin a$

If $a \in a$ then by def of a, $a \notin a$. A contradiction.

If $a \notin a$ then by def of a, $a \in a$. A contradiction. ☒

Thm 5.5 (Non existence of universal set)

There is no a with $a = \{x \,;\, x = x\}$

Proof 1. Proof: Suppose that $a = \{x \,;\, x = x\}$ Then

$\{x \mid x \notin x\} = \{x \in a \mid x \notin x\}$ is a set by Ax of Separation. A contradiction to Thm 5.4.

2. Proof: Suppose that there were a set a with $a = \{x \,;\, x = x\}$ Then $\underline{\mathcal{P}(a) \subseteq a}$.

This is a contradiction to Thm 5.5 (by the following Lemma) ☒

Lemma 5.5 If $X \subseteq Y$ $\underbrace{}_{\text{and } X \neq \emptyset}$, then there is a surjection from Y to X

Proof Let $x^* \in X$. Let $f : Y \to X$ be defined by

$$f(y) = \begin{cases} y & \text{if } y \in X \\ x^* & \text{otherwise.} \end{cases}$$

f is a surjection from Y to X. ☒

Thm 5.6 (Hilbert's Paradox)

There is no set X s.t.

① For any $x \in X$ $\mathcal{P}(x) \in X$

② For any $Y \subseteq X$ $\bigcup Y \in X$

Proof $\bigcup X \in X$ $\mathcal{P}(\bigcup X) \in X$

(If $X \models$ ①② Then)

Then $\mathcal{P}(\bigcup X) \subseteq \bigcup X$

[For any $a \subseteq \bigcup X$, $a \in \mathcal{P}(\bigcup X) \in X$. So $a \in \bigcup X$]

This is a contradiction to Thm 5.3 ☒

Remark

There can be a set X with (a) [$V_\omega \models$ (a)]

$\{\emptyset\} \models$ (b)

Q Is there an infinite set X with $X \models$ (b)?

Grothendieck universe

$H(\kappa)$ for inaccessible κ