__Recap:__

Zermelo's set Theory

__Thm 6.1__ (Z, Zermelo) TFAE

(a) AC

(b) Well-ordering Theorem (Wohlordnungssatz)

(i.e. on every set there is a well ordering)

A pair $\langle X, R\rangle$ with $R \subseteq X^2 (= X \times X)$ is a partial ordering $\Leftrightarrow$ irreflexible (i.e. $a \not R a$ for all $a \in X$)

anti-symmetric (i.e. $aRb \to b \not R a$ for all $a, b \in X$)

$\dfrac{aRb}{a \not R b}$ stands for $\langle a,b\rangle \in R$ — a is smaller than b wrt R

stands for $\langle a,b\rangle \notin R$

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and transitive ie aRb and bRc

Then aRc for all $a, b, c \in X$)

A $\underbrace{\text{partial ordering}}_{\langle X, R\rangle}$ is linear if

$aRb \lor a = b \lor bRa$ holds always for $a, b \in X$

A linear ordering $\langle X, R\rangle$ is a well-ordering if for any $Y \subseteq X$ $Y \neq \emptyset$ there is the minimal element of Y $\oslash$

$\oslash$ wrt R i.e. such $a \in Y$ that for any $b \in Y$ if $b \neq a$ then aRb

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For a linear ordering $\langle X, R\rangle$ $Y \subseteq X$ is said to be an initial segment if Y is closed downward wrt R (i.e., for any $b, c \in X$, if $b \in Y$ and cRb then $c \in Y$)

If $\underbrace{\langle X_0, R_0\rangle}_{X_0''}$ and $\underbrace{\langle X_1, R_1\rangle}_{X_1''}$ are linear orderings

X_1 is an end extension of X_0 if X_0 is an initial segment of X_1 wrt R_1 and $R_0 = R_1 \restriction X_0$

$\boxed{R \restriction X = R \cap X^2}$

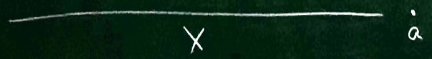
__Lemma 7.1__ (1) If $\langle X, R\rangle$ is a linear ordering (well-ordering resp.) and $Y \subseteq X$

$\langle Y, R \restriction Y\rangle$ is a lin. ordering (well-ordering resp.). If Y is an initial segment of X then $\langle X, R\rangle$ is an end extension of $\langle Y, R \restriction Y\rangle$

(2) If $\langle X, R\rangle$ is a linear ordering (well-ordering resp.) and $a \notin X$ then $\langle X', R'\rangle$ linear ordering (well-ordering resp.) $\oslash$

(2) where $X' = X \cup \{a\}$

$R' = R \cup (X \times \{a\})$.



And $\langle X', R' \rangle$ is an end-extension of $\langle X, R \rangle$

(3) Suppose that $\mathcal{Y}$ is a set consisting of linear orderings (well-orderings resp.) s.t. for any $\langle \underbrace{X_0}_{X_0'}, R_0 \rangle, \langle \underbrace{X_1}_{X_1''}, R_1 \rangle \in \mathcal{Y}$

Either X_1 is an end extension of X_0 or X_0 is an end extension of X_1

Then $\langle X^*, R^* \rangle$ is a linear ordering (well-ordering resp.) where

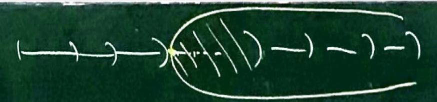
$$X^* = \bigcup \{ \underbrace{X}_{\in \bigcup(\bigcup \mathcal{Y})} \mid \langle X, R \rangle \in \mathcal{Y} \text{ for some } R \}$$

$$R^* = \bigcup \{ \underbrace{R}_{\in \bigcup(\bigcup \mathcal{Y})} \mid \langle X, R \rangle \in \mathcal{Y} \text{ for some } R \}$$

$\mathcal{Y} \ni \langle X, R \rangle = \{\{X\}, \{X, R\}\}$

$\bigcup \mathcal{Y} \ni \{X\}, \{X, R\}$

$\bigcup(\bigcup \mathcal{Y}) \ni X, R$



$\langle X^*, R^* \rangle$ is an end extension of each of $\langle X, R \rangle \in \mathcal{Y}$ ⊠

For a partial ordering $\langle P, R \rangle$ $C \subseteq P$ is a chain if $\langle C, R \restriction C \rangle$ is a linear ordering

Zorn's Lemma: For any partial ordering

" $\langle P, R \rangle$ if any chain C in P has an upper bound in P wrt R

Then there is at least one maximal element of P wrt R

Thm 7.2 (Z, Zorn-(Kuratowski)) TFAE

(a) AC

(b) Zorn's Lemma

For a partial ordering $\langle P, R \rangle$ and $X \subseteq P$, $a \in P$ is an upper bound of X if $x R a$ or $x = a$ holds for any $x \in R$

$a \in P$ is maximal if $a \not R b$ for any $b \in P$

__proof__ By Thm 6.1 it is enough to prove Zorn's Lemma $\Leftrightarrow$ Well-ordering Theorem

"$\Leftarrow$": Assume the Well-ordering Theorem and let $\langle \underline{P}, < \rangle$ be a partial ordering s.t. any chain in $\underline{P}$ has an upperbound.

Let R be a well-ordering on $\underline{P}$

$$\mathfrak{P} = \{ u \in \mathcal{P}(\underline{P}) \mid {<}\restriction u \text{ is a well-ordering on } u, \text{ for any } a \in u \; a \text{ is the minimal element w.r.t. } R \text{ of } \{ b \in \underline{P} \mid b > c \text{ for all } c \in u \text{ with } c < a \} \}$$

As in the proof of Thm 6.1 we can show that for each $u, v \in \mathfrak{P}$ either $\langle u, {<}\restriction u \rangle$ is an end-extension of $\langle v, {<}\restriction v \rangle$ or $\langle v, {<}\restriction v \rangle$ is an end-extension of $\langle u, {<}\restriction u \rangle$.

By Lemma 3.1 (3), letting $u^* = \bigcup \mathfrak{P}$, $\langle u^*, {<}\restriction u^* \rangle$ is a well-ordering and satisfies the condition in the def. of $\mathfrak{P}$. Thus $u^* \in \mathfrak{P}$

Let a^* be an upperbound of u^*. Then $a^* \in u^*$

[Otherwise $\{ b \in \underline{P} \mid c < b \text{ for all } c \in u^* \} \neq \emptyset$. Hence there is the minimal element b^* of the set w.r.t. R. But then $u^{**} = u^* \cup \{ b^* \} \in \mathfrak{P}$, a contradiction to the maximality of u^*]

a^* is maximal in $\langle \underline{P}, < \rangle$ otherwise we can find again an end extension of u^* which should be an element of $\mathfrak{P}$.

Thus this a^* is as desired.

"$\Rightarrow$" Assume that Zorn's Lemma holds.

Let X be a set. Let

$$\mathcal{Y} = \{ \langle u, r \rangle \in \mathcal{P}(X) \times \mathcal{P}(X^2) : r \subseteq u^2 \text{ and } r \text{ is well-ordering of } u \}$$

For $\langle u, r \rangle, \langle u', r' \rangle \in \mathcal{Y}$ let

$\langle u, r \rangle \sqsubset \langle u', r' \rangle \Leftrightarrow$ iff $\langle u', r' \rangle$ is a proper end-extension of $\langle u, r \rangle$

In $\langle \mathcal{Y}, \sqsubset \rangle$ any chain C has its least upperbound $\langle u_C, r_C \rangle$ where

$u_C = \bigcup \{ u \mid \langle u, r \rangle \in C \text{ for some } r \}$

$r_C = \bigcup \{ r \mid \langle u, r \rangle \in C \text{ for some } u \}$

by Lemma 7.1 (3)

Let $\langle u^*, r^* \rangle$ be a maximal element of $\mathcal{Y}$.

We are done by showing

Claim $u^* = X$

$\vdash$ otherwise let $a \in X \setminus u^*$ and let

$u^{**} = u^* \cup \{a\}$ and $r^{**} = r \cup u^* \times \{a\}$

Then $\langle u^{**}, r^{**} \rangle$ is a well-ordering and a proper end-extension of $\langle u^*, r^* \rangle$ by Lemma 7.1 (2)

A contradiction! $\boxtimes$

Thm 7.6 Hilbert's Paradox

There is no set X s.t.

① For any set $x \in X$ $\mathcal{P}(x) \in X$

② For any set $Y \subseteq X$, $\bigcup Y \in X$

There are sets X with $X \models ①$:

for any limit γ, $V_\gamma \models ①$

[If $x \in V_\gamma$ then there is $\alpha < \gamma$ s.t. $x \in V_\alpha$

$x \subseteq V_\alpha$ $\mathcal{P}(x) \subseteq \mathcal{P}(V_\alpha) = V_{\alpha+1}$ $\mathcal{P}(x) \in \mathcal{P}(V_{\alpha+1}) = V_{\alpha+2} \subseteq V_\gamma$]

For any set X

[redacted] $\mathcal{P}(X) \models ②$

$I \subseteq \mathcal{P}(X) \Rightarrow \bigcup I \subseteq X$

$\Leftrightarrow \bigcup I \in \mathcal{P}(X)$

[redacted]

$V \models ①,②$

Lemma 7.3 There are proper subclass X of V with $X \models ①,②$

Proof $X = \{a \mid \text{for any } b \in \mathrm{trcl}(a)\ |b| \leq |a|\} \models ①②$

$X = \{V_\alpha \mid \alpha \in On\} \models ①②$ ☒

Prop 7.4 If a class X is transitive and $X \models ①②$ Then $V = X$

Proof We prove that $V_\alpha \in X$ for all $\alpha \in On$. Since X is transitive it follows that $\underbrace{\bigcup_{\alpha \in On} V_\alpha}_{= V} \subseteq X$, by ind. on α

$\emptyset = \bigcup \emptyset$ $(\emptyset \subseteq X)$ Then by ② $\underbrace{\emptyset}_{= V_0} \in X$

If $V_\alpha \in X$ then $V_{\alpha+1} = \mathcal{P}(V_\alpha) \in X$ by ①

If γ is a limit and we know $V_\alpha \in X$ for all $\alpha < \gamma$

Then $\bigcup \{V_\alpha \mid \alpha < \gamma\} = V_\gamma \in X$. ☒

Thm 7.7 U is weakly G.u. iff $U = H(\kappa)$ with $\kappa = On \cap U$ and κ is regular.

Cor 7.5

If $X \models ①,②$ Then $V_\alpha \in X$ for all $\alpha \in On$ ☒

A set U is a (weakly) Grothendieck universe if ① U is transitive,

② $\emptyset \in U$ this follows from other properties

③ If $a, b \in U$ then $\{a,b\} \in U$

④ If $a \in U$ then $\mathcal{P}(a) \in U$ $\mathcal{P}(a) \subseteq U$

⑤ For any $I \in U$ and $f \in {}^I U$

$\bigcup_{i \in I} f(i) \in U$

Thm 7.6 $\boxed{\text{ZFC}}$ Suppose that U is a G.u. with $\kappa = \mathrm{Ord} \cap U$. Then κ is a regular cardinal, $U = H(\kappa)$ and κ is either $= \omega$ or inaccessible