Formal proof system K^*

"$\to$", "$\neg$", "$\exists$"

binary relation

$\mathcal{L}_\varepsilon = \{\varepsilon\}$

Variables

Quantifiers symbols

$\mathcal{L}$ { Constant symbols, Function symbols, relation symbols

$\varphi_0 \to \varphi_1 \to \varphi_2 \to \cdots \to \varphi_i : (\varphi_0 \to (\varphi_1 (\cdots (\varphi_{m-1} \to \varphi_n)) \cdots))$

Term $\mathcal{L}$ Fmla $\mathcal{L}$

a tautology in the propositional logic

Axiom of K^* Tautology $\psi(\varphi_0, \ldots, \varphi_{m-1})$

Axioms of equality

Axiom of existence $(\varphi(t/x) \to \exists x \varphi)$ if t is substitutable

Reduction Rules

(A) $\dfrac{\varphi \quad (\varphi \to \psi)}{\psi}$ modus ponens

(B) $\dfrac{\varphi \to \psi}{\exists x \varphi \to \psi}$ x is not free in ψ

For a language $\mathcal{L}$, set Γ of $\mathcal{L}$-formulas and an $\mathcal{L}$-formula φ Γ proves φ in K^*

(notation: $\Gamma \vdash \varphi$ or $\Gamma \vdash^{K^*} \varphi$)

- if there are $\mathcal{L}$-formulas $\eta_0, \ldots, \eta_{m-1}, \eta_m$ s.t. $\eta_m = \varphi$ each η_i is either one of the axioms of K^* or an element of Γ or there are $i_0, i_1 < i$ s.t. $\dfrac{\eta_{i_0} \quad \eta_{i_1}}{\eta_i}$ is of the form (A) or there is $i_0 < i$ s.t. $\dfrac{\eta_{i_0}}{\eta_i}$ is of the form (B)

A proof of φ from Γ (in K^*)

If $\mathcal{P}$ is a proof of φ from Γ then we write

$\Gamma \vdash_{\mathcal{P}} \varphi$

Since "proof Theory" in K^*

Lemma 11.1 If $T \vdash \varphi_k \to \varphi_1 \to \cdots \to \varphi_k \to \varphi$ (*)

and $T \vdash \varphi_0$, $T \vdash \varphi_1$ ... $T \vdash \varphi_k$ (**)

then we have $T \vdash \varphi$

- **proof** Suppose $\vec{P} = \langle \eta_0 \ldots \eta_n \rangle$ is a proof of (*) and $\vec{P}_0, \ldots, \vec{P}_k$ are proofs of (**)

Then $\vec{P}^\frown \vec{P}_0^\frown \cdots \vec{P}_k^\frown \langle \varphi_1 \to \cdots \to \varphi_k \to \varphi \rangle \langle \varphi_2 \to \cdots \to \varphi_k \to \varphi \rangle \cdots \langle \varphi_k \to \varphi \rangle^\frown \langle \varphi \rangle$

includes φ_0 $\varphi_0 \to \cdots \to \varphi_k \to \varphi$

This is a proof of φ from T

$$\frac{\varphi_0 \quad (\varphi_0 \to (\varphi_1 \cdots \to \varphi))}{\varphi_1 \to \cdots \to \varphi}$$

$\boxtimes$

Thm 11.2 (Deduction Theorem)

(1) For any (set of) $\mathcal{L}$-formulas T and $\mathcal{L}$-formulas φ, ψ

If $T \vdash \varphi \to \psi$ then $T, \varphi \vdash \psi$

(2) For T and φ, ψ as in (1) if φ is an $\mathcal{L}$-sentence then

$$T \vdash \varphi \to \psi \iff T, \varphi \vdash \psi$$

$T, \varphi \vdash \ldots$ denotes $T \cup \{\varphi\} \vdash \ldots$

proof (1): If $T \vdash \varphi \to \psi$ then

$T, \varphi \vdash \varphi \to \psi$ and $T, \varphi \vdash \varphi$

$\langle \varphi \rangle$ is a proof

Thus by Lemma 11.1 we have $T, \varphi \vdash \psi$

(2): We have to show "$\Leftarrow$"

(†) $T, \varphi \vdash_P \psi \Rightarrow$ there is a proof Q with

$$T \vdash_Q \varphi \to \psi$$

by induction on the length n of P

Truth of formulas in P

If $n = 1$ ψ is either a logical axiom or ① a formula in T or ② φ

If ① or ②

Since $(\psi \to (\varphi \to \psi))$ is a tautology $\langle \psi, (\psi \to (\varphi \to \psi)), (\varphi \to \psi) \rangle$ is a proof of $\varphi \to \psi$ from T

If ③ then $\varphi \to \psi$ is $\psi \to \psi$ which is a tautology so $\langle \psi \to \psi \rangle$ is proof of $\psi \to \psi$ from T

Assume that (†) holds for all proofs of length $< n$ and consider

$T, \varphi \vdash_P \psi$ where P has the length n

Case I ψ is either a logical axiom or one of the formulas among T, φ

Then we can find a proof (of ψ) of length 1 from T, φ.

Case II ψ is deduced by modus ponens from some earlier formulas in P

Exercise

Case III ψ is of the form $\underline{\exists x\, \eta \to \xi}$

and there is an earlier formula in P of the form $\underline{\eta \to \xi}$ ← x is not free in ξ

Since $T, \varphi \vdash_{P'} \eta \to \xi$ where P' is a proper initial segment of P by ind. hyp. $T \vdash \varphi \to \eta \to \xi$

Since $(\varphi \to \eta \to \xi) \to (\eta \to \varphi \to \xi)$ is a tautology we get

$$T \vdash \eta \to \varphi \to \xi$$

By (B)

$$T \vdash \exists x\, \eta \to \varphi \to \xi$$

x is not free here since ψ is a sentence

x is not free

Since $(\exists x\, \eta \to \varphi \to \xi) \to (\varphi \to \exists x\, \eta \to \xi)$ is a tautology,

We get $T \vdash \varphi \to \underbrace{\exists x\, \eta \to \xi}_{\psi}$ ⊠

Work in (a weak fragment Z_0 of) Z

↑ Zermelo's Set Theory

We consider elements of $\omega \times \omega$ as symbols of the logic.

- $\omega \times \{0\}$ $[\langle 0,0\rangle \langle 1,0\rangle \ldots]$ inst. of variables
- $\langle 0,1\rangle, \langle 1,1\rangle, \langle 2,1\rangle, \ldots$: "→", "¬", "∃", "(", ")", ","
- $\omega \times \{2\}$ constant symbols
- $\omega \times \{3\}$ function symbols
- $\omega \times \{4\}$ relation symbols

For $S \subseteq \omega \times \{2,3,4\}$ with arity: $S \cap (\omega \times \{3,4\}) \longrightarrow \omega$

$$Seq = {}^{\omega>}(\omega \times \{0,1,2,3,4\})$$

Then we define

$Term^S \subseteq Seq$

$Fml^S \subseteq Seq^S$

corresponding to the "interpretation" given.

$f : n \to S$ is considered to be a (finite sequence)

identifies f with $f(0), f(1), \ldots, f(n-1)$

$\ulcorner ZFC \urcorner$ $\ulcorner Z_0 \urcorner$ subset of Seq which correspondes axioms of ZFC.

in notation instantiation

If φ is a formula in ZFC

then there is a term $\ulcorner \varphi \urcorner$ of sort (sequence) corresponding to the formula $(ZFC \vdash \ulcorner \varphi \urcorner \in Seq)$

(we expand the language $\mathcal{L}_{ZF}$ to

$\mathcal{L}_{ZF^{\{\}}} = \{ \underset{\text{binary relation symbol}}{\in}, \underset{\text{constant symbol}}{\phi}, \underset{\text{binary function symbols}}{\{\cdot,\cdot\}} \}$

$Z_0^{\{\}}$

$ZFC^{\{\}}$

$ZFC^{\{\}} \vdash \ulcorner \varphi \urcorner \in \ulcorner ZFC \urcorner$

$Z_0^{\{\}}$

also if ψ is a (real) seq of symbols which is not an axiom of ZFC then

$\{\{\phi\}, \phi\} \ldots$ $ZFC^{\{\}} \vdash \neg \ulcorner \psi \urcorner \in \ulcorner ZFC \urcorner$

$Z_0^{\{\}}$

$ZFC^{\{\}} \vdash \exists x (x \in \ulcorner ZFC \urcorner \ldots)$

does not quantifier there is no concrete formula φ with ...