Some more "proof theory" of K^*

A collection of $\mathcal{L}$-sentences T is called a ($\mathcal{L}$-) theory.

> Note: ϕ (empty theory) is also an $\mathcal{L}$-theory

An $\mathcal{L}$-theory T is said to be inconsistent if T proves everything i.e. if for any $\mathcal{L}$-formula φ we have

$$T \vdash^{K^*} \varphi$$

| Otherwise T is consistent

Lemma 12.2 For any language $\mathcal{L}$, and $\mathcal{L}$-theory T TFAE:

(a) T is inconsistent

(b) For some $\mathcal{L}$-formula we have $T \vdash \varphi$ and $T \vdash \neg\varphi$

(c) For some φ: $T \vdash \varphi \wedge \neg\varphi$

(d) $T \vdash x \not\equiv x$

Proof (a) $\Rightarrow$ (b): trivial by def of inconsistency

(b) $\Leftrightarrow$ (c): By Lemma 12.1

(c) $\Rightarrow$ (d) Assume that $T \vdash \varphi \wedge \neg\varphi$ for some (↗)

Lemma 12.1 For any $\mathcal{L}$-formula φ, ψ and $\mathcal{L}$-theory T

$$T \vdash \varphi \quad T \vdash \psi \quad \Leftrightarrow \quad T \vdash (\varphi \wedge \psi)$$

Proof Exercise (Note: $(\varphi \wedge \psi)$ is an abbreviation of $\neg(\varphi \to \neg\psi)$ $\dashv$

(↗) $\mathcal{L}$-formula φ. Since $(\varphi \wedge \neg\varphi) \to x \not\equiv x$ is a tautology Hence $T \vdash (\varphi \wedge \neg\varphi) \to x \not\equiv x$. By Thm 11.2 (Deduction theorem)

$$T \vdash x \not\equiv x$$

(↗)

$((A \wedge \neg A) \to B)$ is a tautology in sentential logic

(↗) (d) $\Rightarrow$ (a): Assume $T \vdash x \not\equiv x$. Since $x \equiv x$ is one of the Axioms of Equality, we have $T \vdash x \equiv x$. By Lemma 12.1 $T \vdash (x \equiv x \wedge \overset{\neg x \equiv x}{x \not\equiv x})$ For any $\mathcal{L}$-formula φ, since $(x \equiv x \wedge x \not\equiv x) \to \varphi$ is a tautology $T \vdash (x \equiv x \wedge x \not\equiv x) \to \varphi$ By Thm 11.1

$$T \vdash \varphi. \quad \boxtimes$$

Objective We introduce the notion of validity of a formula φ $= \varphi(x_0, \dots, x_{n-1})$ in a structure $\mathfrak{A} = \langle A, \dots \rangle$ with parameters $a_0 \dots a_{n-1}$ (notation: $\mathfrak{A} \models \varphi(a_0, \dots, a_{n-1})$)

With this notion, we introduce the semantical deduction by

$$\underset{\substack{\uparrow \\ \mathcal{L}\text{-theory}}}{T} \models \underset{\substack{\uparrow \\ \mathcal{L}\text{-formula}}}{\varphi} \iff \mathfrak{A} \models \varphi \text{ for any } \mathcal{L}\text{-structure } \mathfrak{A} \text{ s.t. } \mathfrak{A} \models \psi \text{ for all } \psi \in T$$

If $\varphi = \varphi(x_0, \dots, x_{n-1})$ we consider the universal closure $\forall x_0 \dots \forall x_{n-1} \varphi$ and consider $\mathfrak{A} \models \varphi$ as an abbreviation of

$$\mathfrak{A} \models \underbrace{\forall x_0 \dots \forall x_{n-1} \varphi}_{\mathcal{L}\text{-sentence}}$$

$$\boxed{\begin{array}{l} \text{A form of Completeness Theorem} \\ \qquad T \models \varphi \iff T \vdash \varphi \end{array}}$$

Lemma 12.3 For any $\mathcal{L}$-formula φ and $\mathcal{L}$-theory T

$$T \vdash \varphi \iff T \vdash \forall x_0 \dots \forall x_{n-1} \varphi$$

proof It is enough to show:

$$T \vdash \varphi \iff T \vdash \forall x \varphi$$

($\Rightarrow$): Suppose $T \vdash \varphi$. Let $\boxed{\psi}$ be any $\mathcal{L}$-sentence which is a tautology. (e.g. $(\exists x\, x=x \to \exists x\, x=x)$ will do)

Since $T, \psi \vdash \varphi$ we have $T \vdash \psi \to \varphi$ by Thm 11.1.

Since (⊛) $(\psi \to \varphi) \to (\neg\varphi \to \neg\psi)$ is a tautology

$T \vdash$ ⊛

Hence by Deduction Theorem

$T \vdash \neg\varphi \to \neg\psi$ by (B). (Existentiation Rule)

$T \vdash \exists x \neg\varphi \to \neg\psi$

Now by the tautology

$$(\exists x \neg\varphi \to \neg\psi) \to (\psi \to \neg\exists x \neg\varphi)$$

We obtain

$$\left.\begin{array}{l} T \vdash \psi \to \underbrace{\neg\exists x \neg\varphi}_{\forall x \varphi} \\ T \vdash \psi \text{ since } \psi \text{ is a tautology} \end{array}\right\} \Rightarrow T \vdash \forall x \varphi$$

"$\Leftarrow$": Assume $T \vdash \forall x \varphi$. So let ψ be again an $\mathcal{L}$-sentence which is a tautology

$\neg \exists x \neg \varphi$

We have

$T, \psi \vdash \neg \exists x \neg \varphi$

Hence, by Deduction Theorem, $T \vdash \psi \to \neg \exists x \neg \varphi$

Since $(\psi \to \neg \exists x \neg \varphi) \to (\exists x \neg \varphi \to \neg \psi)$ is a tautology

$T \vdash \exists x \neg \varphi \to \neg \psi$

By Axiom of Substitution $T \vdash \neg\varphi \to \exists x \neg\varphi$

Hence by the tautology

$(\neg \varphi \to \exists x \neg \varphi) \to (\exists x \neg \varphi \to \neg \psi) \to (\psi \to \varphi)$

$T \vdash \psi \to \varphi$ Since $T \vdash \psi$ it follows that

$T \vdash \varphi$. ☒

Axiom of Substitution

$\varphi(t/x) \to \exists x \varphi$

$(\neg A \to B) \to (B \to \neg C) \to C \to B$

Thm 12.4 (Substitution Theorem)

(*) $T \vdash \varphi$ and if $\mathcal{L}$-term t is substitutable in x in φ then

$T \vdash \varphi(t/x)$

proof By assumption t is also substitutable in x in $\neg\varphi$. Thus $\neg\varphi(t/x) \to \exists x \neg \varphi$ is an instance of the Axiom of Substitution and thus

$T \vdash \neg \varphi(t/x) \to \exists x \neg \varphi$.

Since $(\neg\varphi(t/x) \to \exists \neg \varphi) \to (\neg \exists x \neg \varphi \to \varphi(t/x))$ is a tautology, we obtain $T \vdash \neg \exists x \neg \varphi \to \varphi(t/x)$

By Lemma 12.3 (*) implies $T \vdash \neg \exists x \neg \varphi$

Thus $T \vdash \varphi(t/x)$ ☒

Let C be an infinite collection of new constant symbols not in $\mathcal{L}$

Let $\mathcal{L}'$ be $\mathcal{L}$ with these new constant symbols

Lemma 12.5 For an $\mathcal{L}$-theory T and $\mathcal{L}$-formula

$T \vdash \varphi \Leftrightarrow T \vdash \varphi$ with T and φ seen as $\mathcal{L}'$-theory and $\mathcal{L}'$-formula

proof $\Rightarrow$ is trivial

$\Leftarrow$: Suppose that $(\varphi_0, \dots \varphi_n)$ is a proof of φ from T in $\mathcal{L}'$ where $\varphi_n = \varphi$

Let $c_0, \dots c_{m-1}$ be the new constant symbols which appear in this proof. Let $x_0, \dots x_{m-1}$ be variables which do not appear in the proof

By replacing $c_0, \dots, c_{m-1}$ by $x_0, \dots, x_{m-1}$ resp. in the proof we get a proof of φ from T in $\mathcal{L}$.