We work in (a fragment of) Z

For a language $\mathcal{L}$ if c_i $i\in I$, f_j $j\in J$, r_k, $k\in K$ are the non-logical symbols, i.e. constant, function and relation symbols resp. Then we simply write

$$\mathcal{L} = \{c_i, f_j, r_k\}_{i\in I,\, j\in J,\, k\in K}.$$

For a language $\mathcal{L}$ an $\mathcal{L}$-structure is a set of the form $\mathfrak{A} = \langle A, c_i^{\mathfrak{a}}, f_j^{\mathfrak{a}}, r_k^{\mathfrak{a}}\rangle_{i\in I,\, j\in J,\, k\in K}$

s.t. $A \neq \emptyset$, $c_i^{\mathfrak{a}} \in A$ for each $i\in I$

on support of $\mathfrak{A}$

underlying set of of the structure $\mathfrak{A}$

$f_j^{\mathfrak{a}}: A^n \to A$ for $j\in J$ when f_j is an n-ary function symbol. (we assume some mapping $a: J\to\omega$ is fixed once $\mathcal{L}$ is chosen s.t. $a(j)$ gives the arity of the function symbol f_j)

$r_k^{a} \subseteq A^m$ for $k\in K$ when r_k is an m-ary relation symbol. (we assume ....)

For an $\mathcal{L}$-term $t = t(\underbrace{x_0,\dots,x_{l-1}}_{\vec{x}})$ and $\mathcal{L}$-structure $\mathfrak{A} = \langle A, \dots\rangle$

we define recursively $t^{\mathfrak{A}}_{\vec{x}}: A^l \to A$ by:

1) a) if $t_{\vec x} = x_p$ then (for $p<l$)

$$t^{\mathfrak{A}}_{\vec x}(a_0,\dots,a_{l-1}) = a_p \text{ for } a_0,\dots,a_{l-1}\in A$$

b) if $t_{\vec x} = c_i$ for some $i\in I$ then

$$t^{\mathfrak a}_{\vec x}(a_0,\dots,a_{l-1}) = c_i^{\mathfrak a} \text{ for } a_0,\dots,a_{l-1}\in A$$

2) if $t = f_j(t_0, \dots, t_{n-1})$ where $j\in J$, f_j is n-ary and $t_0 = t_0(x_0,\dots,x_{l-1})$ ... $t_{n-1} = t_{n-1}(x_0,\dots,x_{l-1})$ are $\mathcal{L}$-terms and $t^{\mathfrak a}_{0\vec x}$ .... $t^{\mathfrak a}_{n-1\vec x}$ are already defined

Then, let

$$t^a_{\vec x}(a_0,\dots,a_{l-1}) = f_j^{\mathfrak a}(t^a_{0\vec x}(a_0,\dots,a_{l-1}),\dots, t^a_{n-1,\vec x}(a_0,\dots,a_{l-1}))$$

for all $a_0,\dots,a_{l-1}\in A$

For $\mathcal{L}$-formula $\varphi = \varphi(x_0, \ldots, x_{\ell-1})$ and $a_0, \ldots, a_{\ell-1} \in A$ (for an $\mathcal{L}$-structure $\mathfrak{A} = \langle A, \ldots \rangle$)

the relation

$\mathfrak{A} \models \varphi(a_0, \ldots, a_{\ell-1})$

$\uparrow$ models

is defined recursively as follows:

(0) If $t_0 = t_0(x_1, \ldots, x_{\ell-1})$ and $t_1 = t_1(x_0, \ldots, x_{\ell-1})$ are $\mathcal{L}$-terms and φ is the formula, then

$t_0 \equiv t_1$

$\mathfrak{A} \models \varphi(a_0, \ldots, a_{\ell-1}) :\Leftrightarrow t^{\mathfrak{A}}_{0,\vec{x}}(a_0, \ldots, a_{\ell-1}) = t^{\mathfrak{A}}_{1,\vec{x}}(a_0, \ldots, a_{\ell-1})$

(1) If φ is of the form $r_k(t_0, \ldots, t_{m-1})$ where $k \in K$, r_k is an m-ary relation symbol and $t_0, \ldots, t_{m-1}$ are $\mathcal{L}$-terms, then

$\mathfrak{A} \models \varphi(a_0, \ldots, a_{\ell-1}) :\Leftrightarrow$

$\langle t^{\mathfrak{A}}_{0,\vec{x}}(a_0, \ldots, a_{\ell-1}), \ldots, t^{\mathfrak{A}}_{m-1,\vec{x}}(a_0, \ldots, a_{\ell-1})\rangle \in r^{\mathfrak{A}}_k$

(2) (a) If φ is of the form $(\varphi_0 \to \varphi_1)$ then

$\mathfrak{A} \models \varphi(a_0, \ldots, a_{\ell-1}) :\Leftrightarrow \mathfrak{A} \not\models \varphi_0(a_0, \ldots, a_{\ell-1})$ or $\mathfrak{A} \models \varphi_1(a_0, \ldots, a_{\ell-1})$

(b) If φ is of the form $\neg\varphi_0$

$\mathfrak{A} \models \varphi(a_0, \ldots, a_{\ell-1}) :\Leftrightarrow \mathfrak{A} \not\models \varphi_0(a_0, \ldots, a_{\ell-1})$

(3) If φ is of the form $\exists x\, \varphi_0$

(a) x is among $x_0, \ldots, x_{\ell-1}$ and

say $x = x_0$ for simplicity

$\varphi_0 = \varphi_0(x_0, \ldots, x_{\ell-1})$

Then

$\mathfrak{A} \models \varphi(a_0, \ldots, a_{\ell-1})$

$\Leftrightarrow$ there is some $a \in A$ s.t.

$\mathfrak{A} \models \varphi_0(a, a_1, \ldots, a_{\ell-1})$

(b) x is not in the list $x_0, \dots, x_{\ell-1}$

Call x x_ℓ. $\varphi_0 = \varphi_0(x_0, \dots, x_\ell)$

$\mathfrak{A} \models \varphi(a_0, \dots, a_{\ell-1})$

$:\Leftrightarrow \mathfrak{A} \models \varphi_0(a_0, \dots a_{\ell-1}, a)$

for some $a \in A$.

Strictly speaking we have to show the following by induction along with the introduction of $t^{\mathfrak{a}}$ $\mathfrak{a} \models \varphi(\dots)$:

Lemma 13.1 If $\ell < \ell'$ and

(1) $t = t(x_0, \dots x_{\ell-1})$ then for $\vec{x} = \langle x_0, \dots x_{\ell-1} \rangle$ $\vec{x}' = \langle x_0, \dots x_{\ell-1}, \dots x_{\ell'-1} \rangle$

$t^{\mathfrak{A}}_{\vec{x}}(a_0, \dots a_{\ell-1}) = t^{\mathfrak{A}}_{\vec{x}'}(a_0, \dots a_{\ell-1}, \dots a_{\ell'-1})$

(2)

$\mathfrak{A} \models \varphi(a_0 \dots a_{\ell-1}) \Leftrightarrow \mathfrak{A} \models \varphi(a_0, \dots a_{\ell-1}, \dots a_{\ell'-1})$

(0) If φ is an $\mathcal{L}$-sentence then by Lemma 13.1 $\mathfrak{A} \models \varphi$ does not depend on parameters! $\varphi = \varphi(x_0 \dots x_{\ell-1})$

For $\mathcal{L}$-theory T and $\mathcal{L}$-formula

(1) An $\mathcal{L}$-structure $\mathfrak{A}$ is a model of T (notation: $\mathfrak{A} \models T$) if $\mathfrak{A} \models \psi$ for all $\psi \in T$

(2) $T \models \varphi$ if for any $\mathcal{L}$-structure $\mathfrak{A}$ with $\mathfrak{A} \models T$

We have $\mathfrak{A} \models \varphi$

For an $\mathcal{L}$-formula $\varphi = \varphi(x_0, \dots x_{n-1})$ we write $\mathfrak{A} \models \varphi$ for $\mathfrak{A} \models \underbrace{\forall x_0 \dots \forall x_{n-1}}_{\forall\text{-closure}} \varphi$

Objective:

(A form of) The Completeness Theorem

For any $\mathcal{L}$-theory T and $\mathcal{L}$-formula

$T \models \varphi \Leftrightarrow T \vdash^{K^*} \varphi$

This is equivalent to:

Completeness Theorem

For any $\mathcal{L}$-theory T

T is consistent $\Leftrightarrow$ there is a model $\mathfrak{A}$ of T

i.e. an $\mathcal{L}$-structure $\mathfrak{A}$ s.t. $\mathfrak{A} \models T$

Thm (Soundness Theorem) For any $\mathcal{L}$-theory T and $\mathcal{L}$-formula φ

$T \vdash \varphi \Rightarrow T \models \varphi$

Proof By induction on the length of the proof $T \vdash \varphi$ if the proof has length one then either φ is a logical axiom or $\varphi \in T$

If φ is a logical axiom (i.e. φ is either tautology or one of the equality axioms or of the form $\varphi(t/x) \to \exists x \varphi$

Lemma For any $\mathcal{L}$-structure $\mathfrak{A}$ and any logical axiom φ we have $\mathfrak{A} \models \varphi$

Proof Exercise. ∎ ...