Thm 14.1 (Soundness Thm)

For any $\mathcal{L}$-Theory T and $\mathcal{L}$-formula φ

(*) $T \vdash^{K} \varphi \Rightarrow T \models \varphi$

<u>Convention</u> For $\mathcal{L}$-formula $\varphi = \varphi(x_0, \ldots, x_{\ell-1})$

The $\mathcal{L}$-sentence $\forall x_0 \ldots \forall x_{\ell-1} \varphi$ is called universal closure of φ. Lemma 12.3 $T \vdash^{K} \varphi \Leftrightarrow T \vdash^{K} \forall \vec{x} \varphi$

$T \models \varphi$ denotes $T \models \forall \vec{x} \varphi$

$\Leftrightarrow$ For any $\mathcal{L}$-structure $\mathfrak{A}$ with $\mathfrak{A} \models T$ we have $\mathfrak{A} \models \forall \vec{x} \varphi$

$\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{A} \models \forall \vec{x} \varphi$

<u>Proof</u> For $n \in \omega$ we prove

$(*)_n$ If φ is provable from T by a formula of length $\leq n$ then $T \models \varphi$

by induction on $n \in \omega$

<u>Beginning of the Induction</u> $n = 1$ Done last time.

<u>Induction step</u>

It is enough to show

$(\ast\ast)$ If $\mathfrak{A} \models \varphi(\vec{a})$ $\mathfrak{A} \models \varphi \to \psi(\vec{a})$ then $\mathfrak{A} \models \psi(\vec{a})$

and ⓐ

(A) modus ponens

$$\frac{\varphi \quad (\varphi \to \psi)}{\psi}$$

(B) existential rule

$$\frac{(\varphi \to \psi)}{\exists x \varphi \to \psi}$$

x is not free in ψ

ⓐ $(\ast\ast\ast)$ If $\mathfrak{A} \models \varphi \to \psi(a, \vec{a})$ and x is bound in ψ and $\varphi = (x \vec{x})$ and $\psi = (\vec{x})$

then $\mathfrak{A} \models \exists x \varphi \to \psi(\vec{a})$

for some a

Suppose $(*)_m$ holds for all $m < n$. We show $(*)_n$.

$n > 1$

Suppose that there is a proof of φ from T of length $\leq n$. If there is a proof of length $< n$ then by ..., we have $\mathfrak{A} \models \varphi$.

Thus we may assume that the shortest proof of φ is of length n.

Let P be such a proof. The last step of the proof is one of (A) or (B)

<u>Case I</u> there are formulas φ_0, $\varphi_0 \to \psi \in P$ and the last formula of P is φ

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure with $\mathfrak{A} \models T$

Then φ_0 and $\varphi_0 \to \psi$ have both proofs from T of length $< n$. So by the ind. hyp. $\mathfrak{A} \models \varphi_0$, $\mathfrak{A} \models \varphi_0 \to \varphi$. We show $\mathfrak{A} \models \varphi$. Assume $\varphi = \varphi(x_0, \ldots, x_{\ell-1})$ and $a_0 \ldots a_{\ell-1} \in A$. Then $\mathfrak{A} \models \varphi_0(a_0, \ldots, a_{\ell-1})$ and $\mathfrak{A} \models \varphi_0 \to \varphi(a_0, \ldots, a_{\ell-1})$. Then $\mathfrak{A} \models \varphi(a_0, \ldots, a_{\ell-1})$ by $(\ast\ast)$ as desired.

Case II φ is of the form $\exists x \varphi_0 \to \varphi_1$ and $\varphi_0 \to \varphi_1 \in P$. The rest is analogous to Case I using $(\overset{*}{**})$. ⊠

Thm 14.2 (Completeness Theorem)

For any $\mathcal{L}$-theory T and $\mathcal{L}$-formula φ

(0) $T \models \varphi \Rightarrow T \vdash^{K^*} \varphi$

(Thus by Thm 14.1 we have $T \models \varphi \Leftrightarrow T \vdash^{K^*} \varphi$)

Thm 14.3 (Another version of Completeness Theorem)

For any $\mathcal{L}$-theory T, if T is consistent (i.e. not inconsistent) then T has a model $\mathfrak{A}$

$T \nvdash x \neq x$

(i.e. an $\mathcal{L}$-structure $\mathfrak{A}$ with $\mathfrak{A} \models T$ ($\Leftrightarrow \mathfrak{A} \models \varphi$ for all $\varphi \in T$))

First lecture in the second part: June 10.

Report for the grade! Do as much "Exercise" as possible but at least 3. Dead line: June 9. Submit by email

Thm 14.2 $\Rightarrow$ Thm 14.3:

Suppose Thm 14.3 does not hold then there is a consistent theory T without any model of it. Let φ be an $\mathcal{L}$-sentence s.t. $T \nvdash \varphi$. We have $T \models \varphi$. Thus Thm 14.2 does not hold.

Thm 14.3 $\Rightarrow$ Thm 14.2

Suppose Thm 14.2 does not hold.

Then, there is some $\mathcal{L}$-theory T and an $\mathcal{L}$-sentence φ s.t. $T \models \varphi$ but $T \nvdash^{K^*} \varphi$

Then $T \cup \{\neg\varphi\}$ is consistent. [Otherwise $T \cup \{\neg\varphi\} \vdash \varphi$. By Deduction Thm $T \vdash \neg\varphi \to \varphi$. Since $(\neg\varphi \to \varphi) \to \varphi$ is a tautology $T \vdash (\neg\varphi \to \varphi) \to \varphi$. Thus $T \vdash \varphi$. A contradiction.]

By Thm 14.3, there is a $\underbrace{\text{model}}_{\mathfrak{A}}$ of $T \cup \{\neg\varphi\}$

This is a contradiction to $T \models \varphi$. ⊠

We prove Thm 14.3:

We may assume that there are infinitely many constant symbols which do not appear in $\mathcal{L}$.

Let C be (countably) infinite set of constant symbols not used in $\mathcal{L}$.

Let $\mathcal{L}_C = \mathcal{L} \cup C$

Lemma 14.4 If T is a consistent $\mathcal{L}$-theory then there is a consistent $\mathcal{L}_C$-theory T_C s.t.

(1) $T \subseteq T_C$

(2) for any $\mathcal{L}_C$-sentence φ $\varphi \in T_C$ or $\neg\varphi \in T_C$

(3) for any $\mathcal{L}_C$-sentence φ if $T_C \vdash \varphi$ then $\varphi \in T_C$

(4) If $\exists_x \varphi \in T_C$ then there is $c \in C$ s.t. $\varphi(c/x) \in T_C$

T_C as above is called a Henkin theory (over T)

Proof Since $\mathrm{Fml}_{\mathcal{L}_C}$ is countable we can enumerate them by φ_n, $n \in \omega$. We assume that the enumeration is chosen so that for each $\mathcal{L}_C$-formula φ $\{n \in \omega \mid \varphi_n = \varphi\}$ infinite

We define a sequence $\{\psi_n ; n \in \omega\}$ of $\mathcal{L}_C$-sentences as follows. Letting $T_n = T \cup \{\psi_k \mid k < n\}$, we want that $T_C = \bigcup_{n \in \omega} T_n$ is as desired.

(5) If φ_n is an $\mathcal{L}_C$-sentence then ψ_n is one of φ_n or $\neg\varphi_n$ s.t. T_{n+1} is consistent.

(6) If $\varphi_n = \varphi_n(x)$ and $\exists_x \varphi_n \in T_n$ then $\psi_n = \varphi(c/x)$ where c is the first constant symbol in C which does not appear in T_n

(7) $\psi_n = \exists_x\, x = x$ otherwise.

Let $T_C = \bigcup_{n \in \omega} T_n$

Claim T_C is a Henkin theory over T.

$\vdash$ (1): $T \subseteq T_C$ Clear by $T \subseteq T_n$ by def.

(2): holds by (5)

Suppose φ is an $\mathcal{L}_C$-sentence then $\varphi = \varphi_n$ for some $n \in \omega$ then $\varphi_n \in T_{n+1} \subseteq T$ or $\neg\varphi_n \in T_{n+1} \subseteq T$

T_C is consistent by Claim 1.

If $T_0 \subseteq T_1 \subseteq \dots$ are consistent then $\bigcup_{n \in \omega} T_n$ is also consistent

Claim 1 each T_n is consistent and it is possible to take ψ_n's as in (5), (6), (7)

For (5), If T_n is consistent then $\underbrace{\text{one}}_{\text{at least}}$ of $T_n \cup \{\varphi_n\}$ or $T_n \cup \{\neg\varphi_n\}$ is consistent

[Otherwise $T_n, \neg\varphi_n \vdash x \neq x$ $T_n, \varphi_n \vdash x \neq x$.

But then by Deduction Thm. $T_n \vdash \neg\varphi_n \to x \neq x$ $T_n \vdash \varphi_n \to x = x$

Since $(\varphi \to \varphi_1) \to (\neg\varphi \to \varphi_1) \to \varphi_1$ is a tautology it follows from Lemma 11.1 $T_n \vdash \varphi_n$. A contradiction to the assumption

For (6) If T_n is consistent and $\exists_x \varphi_n \in T_n$ then $\{\varphi_n(c/t)\} \cup T_n$ is consistent. (Exercise!)

(3) Suppose φ is an $\mathcal{L}_C$-sentence and $T_C \vdash \varphi$ then there is some $k \in \omega$ s.t. $T_k \vdash \varphi$ let $n > k$ be s.t. $\varphi = \varphi_n$. Then we should have $\psi_n = \underset{\varphi}{\varphi_n}$ by (5) Thus $\varphi \in T_C$

(4) Suppose that $\exists_x \varphi \in T_C$ there is $k \in \omega$ s.t. $\exists_x \varphi \in T_k$

Let $m > k$ s.t. $\varphi_n = \varphi$ $\quad \exists x \varphi_n \in T_k \subseteq T_m$

Then $\varphi(c/x) \in T_{m+1} \subseteq T_c$ by (6). $\boxtimes$

for some c

For $c, d \in C$ let $c \sim d \Leftrightarrow c \equiv d \in T_c$

<u>Claim 1</u> $\sim$ is an equivalence relation

Lemma 14.5 If $T \vdash \varphi(x)$ then

$T \vdash \varphi(t/x)$ $\quad t$: $\mathcal{L}$-term substitutable in φ for x

<u>Proof</u> $\neg\varphi(t/x) \to \exists x \neg\varphi$ an instance of

Axiom of substitution

Thus $T \vdash \neg\varphi(t/x) \to \exists x \neg\varphi$ $\quad$ Thus $T \vdash \neg\exists x \neg\varphi \to \varphi(t/x)$

Since $T \vdash \varphi(x)$ $\quad T \vdash \neg\exists x \neg\varphi$ (Lemma 12.3)

By ____ and ____ it follows $T \vdash \varphi(t/x)$ $\boxtimes$

By Lemma 14.5 and axioms for equality we have

$c \equiv c$

$c \equiv d \to d \equiv c$ $\qquad \in T_c$

$c \equiv d \to d \equiv e \to c \equiv e$

(1) for any $i \in I$ there is $c \in C$ s.t. $c \equiv c_i \in T_c$

<u>Claim 2</u> (2) For $c_0 \dots c_{n-1} \in C$ $\quad f_j, j \in J$

f_j is n-ary there is $c \in C$ s.t.

$$c \equiv f_j(c_0, \dots c_{n-1}) \in T_c$$

<u>Claim 3</u> (1) $c_0, \dots c_{n-1}, c_0' \dots c_{n-1}' \in C$

if $c_0 \equiv c_0', \dots, c_{n-1} \equiv c_{n-1}' \in T_c$

$d \equiv f_j(c_0, \dots, c_{n-1}),\ d' \equiv f_j(c_0', \dots, c_{n-1}') \in T_c$

then $d \equiv d' \in T_c$

(2) For $c_0 \dots c_{n-1}, c_0' \dots c_{n-1}' \in C$ with

$c_0 \equiv c_0', \dots, c_{n-1} \equiv c_{n-1}' \in T_c$ then we have

$$r_k(c_0, \dots, c_{n-1}) \leftrightarrow r_k(c_0', \dots c_{n-1}') \in T_c$$

<u>Proof of Claim 2 (2)</u> By Lemma 14.6 $\boxtimes$

$A = \{[c] \mid c \in C\}$ $\quad [c]$: the equivalence class of c modulo $\sim$

For $c_i, i \in I$ let $c_i^{\mathfrak{A}} = [c]$ where $c \equiv c_i \in T_c$

For $f_j, j \in J$

<u>Lemma 14.6</u>

$\vdash^{K^*} \exists x\, x \equiv f_j(c_0 \dots c_{n-1})$

proof Exercise. $\boxtimes$

$f_j^{\mathfrak{A}}: A^n \to A \quad \langle [c^0], \dots [c^{n-1}] \rangle \mapsto [c]$

where $c \equiv f_j(c^0, \dots, c^{n-1}) \in T_c$

$r_k^{\mathfrak{A}} = \{\langle [c^0], \dots, [c^{n-1}] \rangle \mid r_k(c^0, \dots c^{n-1}) \in T_c\}$

Let $\mathfrak{A} = \langle A, c_i^{\mathfrak{A}}, f_j^{\mathfrak{A}}, r_k^{\mathfrak{A}} \rangle_{i \in I, j \in J, k \in K}$.

Lemma $\mathfrak{A} \models \varphi([c^0], \dots [c^{l-1}]) \Leftrightarrow \varphi(c^0, \dots c^{l-1}) \in T_c$

proof by induction on φ. $\boxtimes$

In particular $\mathfrak{A} \restriction \mathcal{L}$ is an $\mathcal{L}$-structure with $\mathfrak{A} \models T$

$\boxtimes$