Recapitulation and improvements of last lectures:

Lemma 15.2 ZF + $\cancel{DC}$ Assume:

Either DC holds or A has a wellordering

For a binary relation R on a set A is well-founded on A iff there is no seq $\underbrace{\langle a_n : n\in\omega\rangle}_{\text{in } A}$ s.t. $a_{n+1} R a_n$ for all $n\in\omega$.

Rem if A an ordinal e.g. $A=\omega$ then A has a canonical well ordering.

Lemma 15.3 ZF Assume:

Either DC holds or A has a wellordering:

A binary relation R on A is well-founded iff there is $\varphi : A \to On$ s.t.

$\forall a, a' \in A\ (a R a' \to \varphi(a) < \varphi(a'))$

In the following W_0, W_1 are transitive (possibly proper) class models of a sufficiently large fragment Γ of ZF with $W_0 \subseteq W_1$

Lemma 15.4

(1) If φ is Δ_0 (i.e. all quantifiers appearing in φ are bounded, $\underbrace{\exists x\in y \text{ or } \forall x \in y}_{\text{of the form}}$) letting $\varphi = \varphi(x_0, \dots, x_{n-1})$

Then φ is absolute between W_0 and W_1 (i.e. for any $a_0, \dots, a_{n-1} \in W_0$. Then $W_0 \models \varphi(a_0,\dots,a_{n-1}) \Leftrightarrow W_1 \models \varphi(a_0, \dots, a_{n-1})$)

(2) If φ is Σ_1 (i.e. $\varphi = \exists x_0 \dots \exists x_{m-1} \psi$, ψ Δ_0) then φ is upward absolute

(3) Π_1 φ is downward absolute

If φ is Δ_1^Γ (i.e. there are φ_0, φ_1 s.t. $\Gamma \vdash \varphi \leftrightarrow \varphi_0$ (Σ_1), $\Gamma \vdash \varphi \leftrightarrow \varphi_1$ (Π_1))

Then φ is absolute between W_0 and W_1

Lemma 15.5 for W_0, and W_1 as above and (a) $\Gamma \subseteq ZF + DC$

"x is a well-ordering on y" is Δ_1^{Γ}

and hence absolute between W_0, W_1

(b) $\Gamma \subseteq ZF$. For any $A \in W_0$ s.t. A has a well-ordering in W_0 we have for all $R \in W_0$

$W_0 \models R$ is a well-founded on A

$\Leftrightarrow W_1 \models R$ is a well-founded on A.

Lemma 16.1 ZF (Skolem-Normal Form Th)

Suppose that $\mathfrak{A}$ is a rational structure (i.e. no constants nor functions in the structure) in L

Then for any L-formula $\varphi = \varphi(x_0, \ldots, x_{n-1})$

There is an expansion $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ (in the language $\tilde{L}$) and $\forall\exists$-formula φ^* (in $\tilde{L}$) s.t.

$\tilde{\mathfrak{A}} \models \forall x_0 \ldots \forall x_{n-1} (\varphi \leftrightarrow \varphi^*)$

$\mathfrak{A} = \langle A, \ldots \rangle$

Furthermore there is a $\forall\exists$-formula ψ^* in $\tilde{L}$ s.t., for any $\tilde{L}$-structure $\tilde{\mathfrak{B}}$, if $\tilde{\mathfrak{B}} \models \psi^*$ then $\tilde{\mathfrak{B}} \models \forall x_0 \ldots \forall x_{n-1} (\varphi \leftrightarrow \varphi^*)$

Proof Suppose that φ is of the form

$\exists x \forall y \exists z\, \varphi_0$ where φ_0 is q-free

Let $R^{\tilde{\mathfrak{A}}} = \{ \langle a_0, \ldots a_{n-1}, a \rangle \mid a_0 \ldots a_{n-1}, a \in A$

$\mathfrak{A} \models \forall y \exists z\, \varphi_0(a_0, \ldots a_{n-1}, a, y, z) \}$

Let $\tilde{\mathfrak{A}} = \langle \underbrace{A, \ldots}_{\mathfrak{A}}, R^{\tilde{\mathfrak{A}}} \rangle$.

$\varphi^* = \exists x\, R(x_0 \ldots x_{n-1}, x)$

logically φ^* is equivalent to a $\forall\exists$-formula!

$\psi^* = \forall x_0 \ldots \forall x_{n-1} \forall x\, (R(x_0 \ldots x_{n-1}, x) \rightarrow \forall y \exists z (\varphi_0(\ldots y, z)))$

Prop. 14.? ZF + ~~DC~~

Assume either DC holds or A is well-orderable

In W_1, let $\mathfrak{A} = \langle A, E, \dots \rangle$ be an L-structure and φ an L-sentence.

well-founded in W_1

This condition can be removed (see the next lecture)

Then there is an L-structure $\mathfrak{B} = \langle B, E', \dots \rangle$ in W_0 s.t. E' is well founded (in W_0) and

$W_0 \models \mathfrak{B} \models \varphi$.

$W_0 \models \mathbb{P}$ as a set has a well ordering

Proof Wlog let $\mathfrak{A}$ be relational.

By 16.1 we may also assume that φ is $\forall\exists$-formula. Let (in W_0) $\forall x \exists y\, \varphi_0(x, y)$ (φ_0 q-free)

$\mathbb{P} = \{ \langle \mathfrak{B}, f \rangle \mid \mathfrak{B} = \langle B, E^{\mathfrak{B}} \dots \rangle$ is a finite L-structure, $B \subseteq \omega$, $f : B \to \omega_1^{W_1}$

For $\langle \mathfrak{B}, f \rangle, \langle \mathfrak{B}', f' \rangle \in \mathbb{P}$ with $\mathfrak{B}' = \langle B', E^{\mathfrak{B}'} \dots \rangle$, $\mathfrak{B} = \langle B, E^{\mathfrak{B}}, \dots \rangle$

$\langle \mathfrak{B}, f \rangle \leq_{\mathbb{P}} \langle \mathfrak{B}', f' \rangle \iff f \supseteq f'$, $\mathfrak{B} \supseteq \mathfrak{B}'$

③ for all $a \in B'$ there is $b \in B$ s.t. $\mathfrak{B} \models \varphi_0(a, b)$

(*) for $a, b \in B$ $a E^{\mathfrak{B}} b$ then $f(a) < f(b)$

If $\langle \mathfrak{B}_n, f_n \rangle \in \mathbb{P}$ $n \in \omega$ are decreasing wrt $\leq_{\mathbb{P}}$ then Let

$\mathfrak{B} = \bigcup_{n \in \omega} \mathfrak{B}_n$ $\quad \mathfrak{B} \models \varphi$ by

$f = \bigcup_{n \in \omega} f_n$ $\quad \forall x \exists y\, \varphi_0(x, y)$

O_1

$f : B \to \omega_1^{W_1}$, $B = \bigcup_{n \in \omega} B_n$

For any $a, b \in B$ if $a E^{\mathfrak{B}} b$ then $f(a) < f(b)$ by (*).

Hence such $\mathfrak{B}$ is as desired.

This means that we are done by showing that "$\leq_{\mathbb{P}}$ is not well-founded".

Since "..." is absolute between W_0 and W_1 it is enough to show that there is a descending chain in $(\mathbb{P}, \leq_{\mathbb{P}})$ in W_1.

Working in W_1, Let $f : A \to \mathrm{On}$ s.t. $f(a) < f(b)$ for all $a, b \in A$ with $a E b$

Let $A_n \in [A]^{<\aleph_0}$ $\langle A_n \mid n \in \omega \rangle$ is increasing wrt $\subseteq$ and $\forall a \in A_n\, \exists b \in A_{n+1}$ s.t. $\mathfrak{A} \models \varphi_0(a, b)$. Consider $\mathfrak{A}_n = \mathfrak{A} \restriction A_n$, $f_n : f \restriction A_n$

Let $g : \bigcup_{n \in \omega} A_n \to \omega$ be 1-1 and let

$\mathfrak{A}^*_n = g[\mathfrak{A}_n]$ and $f^*_n : g(A_n) \to \omega_1$ corresponding to f_n

Ⓐ Then $\langle \mathfrak{A}^*_n, f^*_n \rangle \in \mathbb{P}$ and they are decreasing wrt $\leq_{\mathbb{P}}$