Modifying the structure $\mathcal{A}^2$

fully of first order

Identify $n\in\omega$ with $c_n:\omega\to\omega;\ k\mapsto n$

and "$\omega \subseteq {}^\omega\omega$"

$\omega_c=\{c_n : n\in\omega\}$

$ap(f,c_n)=f(n)$

$ap(c_m,c_n)=c_m$

$exp(m,n)=m^n$

$$\mathcal{A}^2=\langle {}^\omega\omega,\ \omega(\cdot),\ ap(\cdot,\cdot),\ \cdot+\cdot,\ \cdot\times\cdot,\ exp(\cdot,\cdot),\ p(\cdot),\ <,\ 0,\ 1\rangle$$

$p(n)$ the n th prime number

all functions return value 0 outside the scope of intended interpretation.

In this setting function quantifiers $\exists^1 x$ $\forall^1 y$ etc are just usual quantifiers! Number quantifiers:

$$\exists^0 n\,\varphi \leftrightarrow \exists n(\underline{\omega}(n)\wedge\varphi)$$

$$\forall^0 n\,\varphi \leftrightarrow \forall n(\underline{\omega}(n)\to\varphi)$$

We may now consider $A\subseteq{}^l({}^\omega\omega)$ instead of $A\subseteq{}^k\omega\times{}^l({}^\omega\omega)$

$\subseteq {}^k({}^\omega\omega)$

The modification of $\mathcal{A}^2$ might change the extent of Δ^0_0

$A\subseteq{}^l({}^\omega\omega)$ is Δ^0_0 if A is defined in $\mathcal{A}^2$ by a bounded formula φ in $\mathcal{L}_{\mathcal{A}^2}$

(i.e. $A=\{\vec{w}\in{}^l({}^\omega\omega) : \mathcal{A}^2\models\varphi(\vec{w})\}$)

φ is bounded iff φ in prenex form contains only number quantifiers which are bounded of the form

$$(\exists^0 n<t),\ (\forall^0 n<t).$$

a term in $\mathcal{A}^2$

$AC_\omega({}^\omega\omega)$: Axiom of choice for countable sequences of sets of reals: For any $\langle A_n\mid n\in\omega\rangle$, $\emptyset\neq A_n\subseteq{}^\omega\omega$ $\exists f:\omega\to{}^\omega\omega$ s.t. $f(n)\in A_n$

This much AC is (always!) assumed from now on,

<u>Projective hierarchy</u>

(*) In ${}^{\omega}\omega$ open = F_σ since ${}^{\omega}\omega$ is Baire

$\vec{\Sigma}$ $\vec{\Pi}$

boldface

Σ Π

$A \subseteq {}^{l}({}^{\omega}\omega)$ is $\vec{\Sigma}^1_0$ if A is open.[(*)] For $n \in \omega$

A is $\vec{\Pi}^1_n$ if ${}^{l}({}^{\omega}\omega) \setminus A$ is $\vec{\Sigma}^1_n$

(in particular A is $\vec{\Pi}^1_0$ iff A is a closed set)

A is Σ^1_{n+1} if $A = p(B)$ where B is Π^1_n
projection

$= \{\langle f_1, \dots, f_n\rangle \in {}^{l}({}^{\omega}\omega) \mid \langle f_0, f_1, \dots f_n\rangle \in B \text{ for some } f_0 \in {}^{\omega}\omega\}$

We consider ${}^{\omega}\omega$ under the product topology of the discrete topology of ω.

A basic open set of ${}^{\omega}\omega$ is of the form

$O_p = \{f \in {}^{\omega}\omega \mid p \subseteq f\}$ for each

partial function $p : D \to \omega$ $\quad D \in [\omega]^{<\aleph_0}$

${}^{l}({}^{\omega}\omega)$ is considered under the product topology of ${}^{\omega}\omega$

A basic open set of this topology is of the form

$O_{p_0} \times \cdots \times O_{p_{l-1}}$ $\quad p_0, \dots, p_{l-1}$ as before.

$\vec{\Delta}^1_n = \vec{\Pi}^1_n \cap \vec{\Sigma}^1_n$ for $n \in \omega$

(with $\vec{\Pi}^1_n$ [$\vec{\Sigma}^1_n$] we also denote the set of

all $A \subseteq {}^{l}({}^{\omega}\omega)$ for some l which are $\vec{\Pi}^1_n$ [$\vec{\Sigma}^1_n$]

<u>Thm 19.1</u> (1) The following diagram holds

$$\begin{array}{ccccccccccc}
 & & \vec{\Sigma}^1_0 & & & & \vec{\Sigma}^1_1 & & & & \vec{\Sigma}^1_2 \\
 & \subsetneq & & \subseteq & & \subsetneq & & \subseteq & & \subsetneq & \\
\vec{\Delta}^1_0 & & & & \vec{\Delta}^1_1 & & & & \vec{\Delta}^1_2 & & \cdots \\
 & \subseteq & & \subsetneq & & \subseteq & & \subsetneq & & \subseteq & \\
 & & \vec{\Pi}^1_0 & & & & \vec{\Pi}^1_1 & & & & \vec{\Pi}^1_2
\end{array}$$

(2) All the inclusions in (1) are proper

(3) $\vec{\Delta}^0_1$ is clopen sets.

(4) $\vec{\Delta}^1_1$ is Borel sets.

$A \subseteq {}^{l}({}^{\omega}\omega)$ is said to be projective if A is $\vec{\Sigma}^1_n$ for some $n \in \omega$

<u>Rem</u> $\vec{\Sigma}^1_1$ sets are (historically) called analytic sets.

$\vec{\Pi}^1_1$ " " co-analytic sets.

Caution in the classical setting where we consider $A \subseteq {}^{l}\mathbb{R}$ The definition of $\vec{\Sigma}^1_0$ set should be F_σ

Thm 19.2 $(ZF + AC_\omega({}^\omega\omega))$

$A \subseteq \ell({}^\omega\omega)$ is $\vec{\Sigma}^1_n$ iff

A is Σ^1_n (in the sense of $\mathcal{A}^2$)

over some $a \in {}^\omega\omega$

Hence $A \subseteq \ell({}^\omega\omega)$ is projective iff

A is definable subset of $\ell({}^\omega\omega)$ in $\mathcal{A}^2$

$A \subseteq \ell({}^\omega\omega)$ is projective iff

A is definable subset of $\ell({}^\omega\omega)$ in $\langle \mathcal{H}(\aleph_1), \in \rangle$

Thm 18.1 $(ZF + AC_\omega({}^\omega\omega))$

For any $n \in \mathbb{N}$, $a \in {}^\omega\omega$ and $A \subseteq \ell({}^\omega\omega)$

A is $\Sigma^1_{n+1}(a)$ iff A is definable by a Σ_n-formula φ (in $\mathcal{L}_\in$) in $\langle \mathcal{H}(\aleph_1), \in \rangle$

The following theorems are not to be proved in this course.

Thm 19.4 (2) If $V = L$ then there is Δ^1_2 sets of reals which are not Lebesgue measurable

(3) If a certain large cardinal exists (proper compact is enough!) Then all projective sets are Lebesgue measurable (and much more!)

(1) All Σ^1_1 and Π^1_1 sets are Lebesgue measurable.

Notations:

For $A \subseteq \ell({}^\omega\omega)$, we write

$\vec{w} \in A \iff A(\vec{w})$

For $m_0, \dots, m_n, i \in \omega$

$$\langle\langle m_0, \dots, m_n \rangle\rangle = \underbrace{p(0)^{m_0+1} \times p(1)^{m_1+1} \times \dots \times p(n)^{m_n+1}}_{\text{definable as an } \mathcal{L}_{\mathcal{A}^2}\text{-term}}$$

$$\underbrace{(m)_i = \min\{e \leq m \mid p(i)^{e+2} \text{ does not divide } m\}}_{\text{definable function}}$$

definable by a bounded formula (i.e. Δ^0_0)

We have

$(\langle\langle m_0, \dots, m_n \rangle\rangle)_i = m_i$

$$A_i = \begin{cases} \langle m_0, \dots, m_n \rangle & \text{if } i = \langle\langle m_0, \dots, m_n \rangle\rangle \\ 0 & \text{otherwise} \end{cases}$$

For $x \in {}^\omega\omega$ and $i \in \omega$

$[x]_i(m) = x(\langle\langle i, m \rangle\rangle)$

$= ap(x, \langle\langle i, m \rangle\rangle)$

definable as an $\mathcal{L}_{\mathcal{A}^2}$-term.

For $x \in {}^\omega\omega$ and $m \in \omega$

$x \restriction m = \langle\langle x(0), x(1), \dots, x(m-1) \rangle\rangle$ (definable in $\mathcal{L}_{\mathcal{A}^2}$)

For $\vec{x} \in \ell({}^\omega\omega)$ and $m \in \omega$ with $\vec{x} = \langle f_0, \dots, f_{\ell-1} \rangle$

$\vec{x} \restriction m = \langle f_0 \restriction m, \dots, f_{\ell-1} \restriction m \rangle$.