Recap: projective hierarchy

$A \subseteq \ell(\omega_\omega)$ is $\vec{\Sigma}^1_0$ is A open

A is $\vec{\Pi}^1_n$ $(n \geq 0)$ if

$\ell(\omega_\omega) \setminus A$ is $\vec{\Sigma}^1_n$

(in particular A is $\vec{\Pi}^1_0$ iff A is closed)

A is $\vec{\Sigma}^1_{n+1}$ iff $A = \overset{\text{projection}}{p}(B)$

$= \{\langle x_1, \dots, x_k\rangle \mid \langle x_0, x_1, \dots, x_k\rangle \in B$ for some $x_0 \in {}^\omega\omega\}$

for some $\vec{\Pi}^1_n$ set $B \subseteq {}^{k+1}(\omega_\omega)$

($\vec{\Sigma}$ $\vec{\Pi}$ boldface Σ and Π)

Σ^1_n, Π^1_n and $\Sigma^1_n(a)$ $\Pi^1_n(a)$ are defined in terms of $\mathcal{A}^2$

Remark $\vec{\Pi}^1_n$ are closed wrt projection

$A \subseteq \ell(\omega_\omega)$ is projective if there is $n \in \mathbb{N}$ s.t. A is $\vec{\Sigma}^1_n$.

A is $\vec{\Delta}^1_n$ iff A is $\vec{\Sigma}^1_n$ and $\vec{\Pi}^1_n$

$$\begin{array}{ccccccccccc}
 & & \vec{\Sigma}^1_0 & & & \subsetneq & \vec{\Sigma}^1_1 & & & \subsetneq & \vec{\Sigma}^1_2 \\
\vec{\Delta}^1_0 & \subsetneq & & \subsetneq & \vec{\Delta}^1_1 & \subsetneq & & \subsetneq & \vec{\Delta}^1_2 & \subsetneq & \cdots\cdots \\
 & & \vec{\Pi}^1_0 & & & \subsetneq & \vec{\Pi}^1_1 & & & \subsetneq & \vec{\Pi}^1_2
\end{array}$$

clopen sets

All the inclusions are proper

For $a \in {}^\omega\omega$ $A \subseteq \ell(\omega_\omega)$ is $\Sigma^1_n(a)$ $n \geq 1$

iff there is a $\mathcal{L}_{\mathcal{A}^2}$-formula φ of the form

$$\exists^1 x_1 \forall^2 x_2 \cdots\cdots Q^1 x_n \varphi_0(x_1, \dots, x_n, y_1 \dots y_\ell, z)$$

s.t. $\vec{w} \in A \iff \mathcal{A}^2 \models \varphi(\vec{w}, a)$ i.e. A is definable by φ with the parameter a

where φ_0 contains only number quantifiers.

still to be proved!

Thm 19.2 (ZF + $AC_\omega(\omega_\omega)$) $n \geq 1$

$A \subseteq \ell(\omega_\omega)$ is $\vec{\Sigma}^1_n$ iff

A is $\Sigma^1_n(a)$ for some $a \in {}^\omega\omega$

Cor 20.1 $A \subseteq \ell(\omega_\omega)$ is projective iff A is definable in $\mathcal{A}$ with some parameter.

$AC_\omega({}^\omega\omega)$: there is a choice function for any countable family of non-empty subsets of ${}^\omega\omega$

Also yet to be proved!

Thm 18.1 ($ZF + AC_\omega({}^\omega\omega)$) for $m \in \omega$

$A \subseteq \mathcal{P}({}^\omega\omega)$ is $\Sigma^1_{m+1}(a)$ iff A is Σ_m definable in $\langle H(\aleph_1), \in \rangle$ with a

Cor 20.2 $A \subseteq \mathcal{P}({}^\omega\omega)$ is projective iff A is definable in $\langle H(\aleph_1), \in \rangle$ with a parameter

Notations:

we consider ω as the subset $\{c_n : n \in \omega\}$ where $(c_n : \omega \to \omega ; k \mapsto n)$

$k \mapsto k$'th prime

$$\mathcal{A} = \langle {}^\omega\omega, \omega, ap(\cdot,\cdot), \cdot + \cdot, \cdot \times \cdot, exp(\cdot,\cdot), p(\cdot), <, 0, 1 \rangle$$

For $m_1 \dots m_n \in \omega$

$$\langle\langle m_0, \dots, m_n \rangle\rangle = p(0)^{m_0+1} \times p(1)^{m_1+1} \times \dots \times p(n)^{m_n+1}$$

expressible by $\mathcal{L}_{\mathcal{A}^2}$-term

$(m)_i = \min\{e < m : p(i)^{e+2} \text{ does not divide } m\}$

definable by bounded formula (Δ^0_0)

$$\Delta_i = \begin{cases} \langle\langle m_0, \dots m_n \rangle\rangle & \text{if } i = \langle\langle m_0, \dots m_n \rangle\rangle \\ 0 & \text{otherwise} \end{cases}$$

$\| \; [f]_i = ap(f, \langle\langle i, \cdot \rangle\rangle)$

For $f \in {}^\omega\omega$

$[f]_i : \omega \to \omega ; m \mapsto f(\langle\langle i, m \rangle\rangle) = ap(f, \langle\langle i, m \rangle\rangle)$

expressible by an $\mathcal{L}_{\mathcal{A}^2}$ term

For $f \in {}^\omega\omega$ $m \in \omega$

$$f \restriction m = \langle\langle f(0), \dots, f(m-1) \rangle\rangle$$

For $i < m$

$$(f \restriction m)_i = f(i)$$

Lemma 20.3 (Contraction of bounded quantifiers)

(1) For any $\mathcal{L}_{\mathcal{A}^2}$-formula φ, $\underbrace{\mathcal{L}_{\mathcal{A}^2}\text{-term } t}$, $\vec{w} \in \ell(\omega_\omega)$

$$\mathcal{A}^2 \models (\forall^0 p < t)\, \exists^0 m\, \varphi(p, m, \vec{w}, \dots)$$

$$\Leftrightarrow \mathcal{A}^2 \models \exists^0 m\, \forall^0 p < t\, \varphi(p, (m)_p, \vec{w}, \dots)$$

(2) ...

$$\mathcal{A}^2 \models (\exists^0 m)(\exists^0 n)\, \varphi(m, n, \dots)$$

$$\Leftrightarrow \mathcal{A}^2 \models (\exists^0 m)\, \varphi((m)_0, (m)_1, \dots)$$

The corresponding assertions also hold for dual formulas

Lemma 20.4 (Contraction of ... of second order quantifiers)

$$\mathcal{A}^2 \models (\exists^1 f_0)(\exists^1 f_1)\, \varphi(f_0, f_1, \dots)$$

$$\Leftrightarrow \mathcal{A}^2 \models (\exists^1 f)\, \varphi([f]_0, [f]_1, \dots)$$

Lemma 20.5 ($\mathrm{ZF} + A_\omega(\omega_\omega)$)

(1) $\mathcal{A}^2 \models \forall^0 m\, \exists^1 f\, \varphi(m, f, \dots)$

$\Leftrightarrow \mathcal{A}^2 \models \exists^1 f\, \forall^0 m\, \varphi(m, [f]_m, \dots)$

(2) $\mathcal{A}^2 \models \exists^0 m\, \exists^1 f\, \varphi(m, f, \dots)$

$\Leftrightarrow \mathcal{A}^2 \models \exists^1 f\, \varphi([f]_0(0), [f]_1, \dots)$

Lemma 20.6 $A \subseteq \ell(\omega_\omega)$ is $\Sigma^1_n(a)$

iff there is a q-free formula $\hat{\varphi}$ s.t.

$\vec{w} \in A \Leftrightarrow$

$$\mathcal{A}^2 \models \exists^1 f_1\, \forall^1 f_2 \dots \underset{\substack{\uparrow \\ \forall^1 \text{ or } \exists^1}}{Q^1} f_n\, \underset{\substack{\uparrow \\ \exists^0 \text{ or } \forall^0}}{Q^0} m\, \hat{\varphi}(m, w_1 \restriction m, w_2 \restriction m \dots f_1 \restriction m, f_2 \restriction m \dots a \restriction m)$$

$\vec{w} = \langle w_1 \dots w_l \rangle$

Th 19.2 follows from this lemma!