For $\vec{f} \in {}^{l}(\omega_\omega)$ with $\vec{f} = \langle f_0, \dots, f_{l-1} \rangle$

we denote with $\vec{f} \restriction m$ the sequence

$\langle f_0 \restriction m, \dots, f_{l-1} \restriction m \rangle$, $m \in \omega$

Remark 21.0 $A \subseteq {}^{l}(\omega_\omega)$ is open iff

for any $\vec{f} \in A$ there $m \in \omega$ s.t.

$[\vec{f} \restriction m] \subseteq A$ where $[\vec{f} \restriction m] = \{ \vec{g} \in {}^{l}(\omega_\omega) :$

$\vec{g} \restriction m = \vec{f} \restriction m \}$

Note that

$[\vec{f} \restriction m]$, $m \in \omega$, $\vec{f} \in {}^{l}(\omega_\omega)$ form an open base

of the topology of ${}^{l}(\omega_\omega)$!

Lemma 21.1 $A \subseteq {}^{l}(\omega_\omega)$ is open

iff there is a q-free formula φ in $\mathcal{L}_{\mathcal{A}^2}$ and

$a \in {}^{\omega}\omega$ s.t. (*) $\vec{f} \in A \iff \mathcal{A}^2 \models \exists^0 m\, \varphi(m, \vec{f} \restriction m, a \restriction m)$

Proof Suppose that A is open (wlog $A \neq \emptyset$). By 21.0 there is a

sequence $\vec{p}_i, n_i$, $i \in \omega$ s.t. $\vec{p}_i \in {}^{l}({}^{n_i}\omega)$ ($n_i \leq i$) s.t.

$$\vec{f} \in A \iff \exists i \in \omega\, (n_i \neq 0 \wedge \vec{p}_i \subseteq \vec{f}).$$

(i.e. $A = \bigcup \{ [\vec{p}_i] \mid i \in \omega\ n_i \neq 0 \}$)

Let $a \in {}^{\omega}\omega$ code $\langle \vec{p}_i \mid i < \omega \rangle$ and $\langle n_i \mid i \in \omega \rangle$

e.g. by letting $a(i) = \langle\langle \vec{p}_i, n_i \rangle\rangle$

Let ψ be the q-free formula s.t.

$\exists^0 i\, \psi(i, \vec{f} \restriction i, a \restriction i) \iff$ there is i s.t.

$n_i \neq 0$ and $\vec{f} \restriction i \supseteq \vec{p}_i \restriction n_i$

Then this ψ is as desired

(i.e. satisfies (*))

Suppose now that φ and $a \in {}^{\omega}\omega$ satisfy

(*) for the set $A \subseteq {}^{l}(\omega_\omega)$.

$$A = \bigcup \{ [\vec{f} \restriction m] : \vec{f} \in {}^{\omega}\omega\ m \in \omega$$

$$\mathcal{A}^2 \models \varphi(m, \vec{f} \restriction m, a \restriction m) \}$$

open

$\boxtimes$

Lemma 21.2 (1) Suppose

$$B = \{ \vec{f} \in {}^{m}({}^{\omega}\omega) : \mathcal{A}^2 \models \varphi(\vec{f}, a) \}$$

for an $\mathcal{L}_{\mathcal{A}^2}$-formula φ and $a \in {}^{\omega}\omega$.

Then $\{ \vec{g} \in {}^{1}({}^{\omega}\omega) : \mathcal{A}^2 \models \exists x\, \varphi(x, \vec{g}, a) \}$

$= p(B)$

(2) If $B = \{ \vec{f} \in {}^{1}({}^{\omega}\omega) : \mathcal{A}^2 \models \varphi(\vec{f}, a) \}$

Then $\{ \vec{f} \in {}^{1}({}^{\omega}\omega) : \mathcal{A}^2 \models \neg\varphi(\vec{f}, a) \} = {}^{1}({}^{\omega}\omega) \setminus B$

Lemma 20.6 (ZF + $AC_\omega({}^{\omega}\omega)$) $A \subseteq {}^{1}({}^{\omega}\omega)$ is $\Sigma^1_n(a)$ iff there is a q-free formula $\hat{\varphi}$ in $\mathcal{L}_{\mathcal{A}^2}$ s.t.

$$\vec{f} \in A \iff \mathcal{A}^2 \models \exists^1 f_1 \forall^1 f_2 \ldots Q^1 f_n Q^0 m \; \hat{\varphi}(m, \vec{f} \restriction m, a \restriction m)$$

Def (given in last lecture) For $A \in {}^{1}({}^{\omega}\omega)$

A is Σ^1_0 if A is open

A is Π^1_n if ${}^{1}({}^{\omega}\omega) \setminus A$ is Σ^1_n

A is Σ^1_{n+1} if $A = p(B)$ for a Σ^1_n set B.

Putting Lemma 21.1, 22.2, 20.6 together and remembering the def in terms of Σ^1_n, Π^1_n. We obtain

Theorem 19.2 (ZF + $AC_\omega({}^{\omega}\omega)$)

$A \subseteq {}^{1}({}^{\omega}\omega)$ is Σ^1_n iff A is $\Sigma^1_n(a)$ for some $a \in {}^{\omega}\omega$.

Cor 21.3 $A \subseteq {}^{1}({}^{\omega}\omega)$ is projective (i.e. Σ^1_n (or Π^1_n) for some $n \in \omega$) iff A is definable in $\mathcal{A}^2$ with some parameters

Note that in $\mathcal{A}^2$ sequences $\vec{a} \in {}^{n}({}^{\omega}\omega)$ can be coded by a single $a \in {}^{\omega}\omega$ (e.g. via $[a]_m$)

Consider the following formulas in $\mathcal{L}_{\mathcal{A}^2}$

(we work under ZF + $AC_\omega({}^{\omega}\omega)$)

$\Phi^1(x)$: "$x \in {}^{\omega}\omega$ and x codes an extensional well-founded binary relation E_x on ω"

e.g. the coding may be given by

$$\langle n, m \rangle \in E_x \iff x(\langle\langle n, m \rangle\rangle) = 1$$

$\Phi^1_n(x, f_1, \ldots, f_{n-1})$: $\Phi^1(x) \wedge$

$f_1, \ldots, f_{n-1} \in$ the transitive collapse of $\langle \omega, E_x \rangle$ (M_x)

$\Phi^2(r, p, f)$: $\Phi^1(r) \wedge \Phi(p) \wedge$

$\mathrm{tncol}(\omega, E_r) \subset \mathrm{tncol}(\omega, E_p)$

and $f : \langle \omega, E_r \rangle \xrightarrow{\cong} \langle \omega, E_p \rangle$

corresponds to the embedding of the transitive collapse!

Let φ be $\mathcal{L}_\in$-formula (coded as an element of ω)

$\Phi^3_m(\varphi, p, f_1 \ldots f_{n-1})$: $\Phi^1_n(p, f_1, \ldots, f_{n-1})$

$\wedge\ M_p \models \varphi(f_1 \ldots f_{n-1})$

$\Phi_\varphi(x)$: $\Phi^1(x)$, φ is absolute between M_x and $H(\aleph_1)$

Lemma 21.4 ($ZF + AC_\omega(\omega_\omega)$)

(1) $\Phi^1(x)$ $\Phi^1_n(\ldots)$ $\Phi^2(\ldots$

$\vec{\Phi}_n(\ldots)$ are Π^1_1-relations

(2) $\Phi_\varphi(x)$ is definable as an $\mathcal{L}_{A^2}$-formula and if φ is Σ_k then $\Phi_\varphi(x)$ can be defined as a Π^1_{k+1} formula

Thm 18.1 For $n \in \omega \setminus 1$, $a \in \omega_\omega$

$A \subseteq {}^k(\omega_\omega)$ is $\Sigma^1_{n+1}(a)$ iff A is Σ_n-definable in $\langle H(\aleph_1), \in \rangle$ with the parameter a.

Proof $\Leftarrow$ (The hard direction)

Suppose $A = \{ \vec{f} \in {}^k(\omega_\omega) ; \langle H(\aleph_1), \in \rangle \models \varphi(\vec{f}, a) \}$ (Σ_n)

Consider the $\mathcal{L}_{A^2}$-formula ψ:

$\exists^1 r \left(\Phi_\varphi(r) \wedge \Phi^3(\varphi, r, \vec{f}, a) \right)$

Π^1_{k+1} by Lemma 21.4. Then ψ is Π^1_{k+1} and

$\vec{f} \in A \Leftrightarrow A^2 \models \psi(\vec{f}, a)$.

$\Rightarrow$ By straight forward induction on n! $\blacksquare$